Comparison of the accuracy of numerical integration of second-order differential equations and the corresponding system of first-order differential equations

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Abstract

The paper presents a comparison of the errors of Störmer' s method in the case of a single differential equation and Adams' method in the case of the corresponding system of first-order differential equations. The arguments are conducted within the framework of linearized error theory. The considerations primarily concern the case where the derivative of the right-hand side of the equation with respect to the unknown function is positive. The results of the work are illustrated with examples. Bibliography: 7 items.

Full Text

Preamble

In 1967, L. Ya. Adrianova [**?**] investigated the numerical integration of the second-order differential equation

$$x'' = f(t, x)$$

under the initial conditions $x(t_0) = x_0$ and $x'(t_0) = z(t_0) = x'_0$. This system can be represented as a first-order system of the form:

$$x' = z$$
$$z' = f(t, x)$$

with initial conditions $x(t_0) = x_0$ and $z(t_0) = z_0$.

The numerical solution of this problem is often approached using methods such as those proposed by Runge-Kutta or Störmer. As noted in the literature [**?**, **?**, **?**], the choice of step size h and the order of the method significantly influence the stability and accuracy of the resulting solution. For instance, the work of N. S. Bakhvalov [?] emphasizes the importance of the error term in these approximations. In this section, we analyze the local and global errors associated with these numerical schemes, specifically focusing on the behavior of the solution $x(t)$ and its derivatives up to order $r+1$ within the interval $[t_0, T)$.

1. Linear Error Analysis for Systems (3)-(4)

Consider the discrete points $t_k = t_0 + kh$ for $k = 0, 1, 2, \ldots, N$. Let x_k and z_k denote the numerical approximations of $x(t_k)$ and $z(t_k)$, respectively. The global error at step k is defined by the vector $\Delta_k = (x(t_k) - x_k, z(t_k) - z_k)$. Following the methodology established in [?], the evolution of the error can be described by a linear differential system:

$$\Delta'(t) = A(t)\Delta(t) + h^r[q(t) + p(t)]$$

where $\Delta(t_0) = 0$. Here, $q(t)$ represents the principal part of the local truncation error, typically proportional to $h^{r+1}x^{(r+1)}(t)$, and $p(t)$ accounts for higher-order terms.

The solution to this error equation can be expressed using the fundamental matrix or influence functions $\phi_1(t_0, t)$ and $\phi_2(t_0, t)$. These functions satisfy the homogeneous part of the error equation with initial conditions $\phi_1(t_0, t_0) = 1, \phi_2(t_0, t_0) = 0$ and their respective derivatives. The resulting error at time T is given by the integral representation:

$$\Delta(T) = \int_{t_0}^{T} [\phi_1(\xi, T)q_1(\xi) + \phi_2(\xi, T)q_2(\xi)]d\xi + O(h^{r+1})$$

2. Linear Error Analysis for Equation (1)-(2)

For the second-order equation $x'' = f(t, x)$, the error $\epsilon(t) = x(t) - x_k$ satisfies a similar linear relationship. Using the results from [?], the error at the k-th step can be approximated by:

$$\Delta''(t) = f_x(t)\Delta(t) + h^{r+1}\gamma_{r+1}x^{(r+1)}(t)$$

where $f_x(t)$ is the partial derivative of f with respect to x evaluated along the trajectory. The influence of the initial conditions and the accumulated truncation error leads to the final error expression:

$$\epsilon(T) = h^{r+1}\gamma_{r+1} \int_{t_0}^{T} \Phi(t, \xi)x^{(r+1)}(\xi)d\xi$$

where $\Phi(t, \xi)$ is the Green' s function associated with the linearized operator.

3. Convergence and Stability Results

We compare the error magnitudes between different integration schemes. Let $\epsilon_1(T)$ and $\epsilon(T)$ represent the errors of two distinct methods. Under the assumption that the $(r+3)$-th derivative of the solution is bounded on $[t_0, T]$, we can establish bounds on the relative error. Specifically, if $f_x(t) > 0$, the influence functions ϕ_1 and ϕ_2 are non-negative, which allows for a monotonic characterization of the error growth.

As shown in [**?**], if the condition $x^{(r+2)}(t_0) \neq 0$ holds, the sign of the error is determined by the sign of the higher-order derivatives of the exact solution. We demonstrate that for a class of problems where $f_x(t)$ remains positive, the numerical solution remains stable, and the global error is bounded by:

$$|\epsilon(T)| \leq Ch^{r+1} \max_{t\in[t_0,T]} |x^{(r+1)}(t)|$$

where C depends on the length of the interval and the properties of the function f.

4. Numerical Examples and Discussion

To validate the theoretical findings, we consider several test cases with a step size $h = 0.05$.

Example 1. Consider the equation $x'' = x$ with initial conditions $x(0) = 1, x'(0) = 1$, which has the exact solution $x(t) = e^t$. For $T = 5$ and $N = 100$, the calculated error $\delta_1(5) - \delta(5)$ exceeds 125×10^{-6}. The influence function $\phi_2(\xi, 5) = \sinh(5 - \xi)$ remains positive, confirming the theoretical prediction that the error grows exponentially but remains predictable.

Example 2. For the same equation with $x(0) = 1, x'(0) = -1$, the solution is $x(t) = \exp(-t)$. In this case, the fourth derivative $x^{(4)}(\xi) = 0.5002$ at specific points, and the error magnitude $|\epsilon_1(5)|$ is found to be less than $|\epsilon(5)|$, consistent with the stability criteria for decaying solutions.

Example 3. Consider $x'' = -\frac{x}{9}$ with $x(0) = 0, x'(0) = \frac{1}{3}$, yielding $x(t) = \sin(t/3)$. Over the interval $[0, 4.5]$, the function $\phi_2(\xi, 4.5) = 3\sin(\frac{4.5-\xi}{3})$ is positive. The numerical results show that the error oscillates in accordance with the trigonometric nature of the solution, with the global error at $N = 90$ steps matching the predicted bounds.

Example 4. Finally, we examine a non-autonomous case where $x(t) = t(2 + \ln t)$ on the interval $[1, 6]$. The derivative $x^{(4)}(\xi) \approx -0.07$ remains small, and the numerical error $|\epsilon_1(6)|$ is approximately 3.96×10^{-3}, demonstrating the robustness of the error estimates even for logarithmic growth.

References

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Note: Figure translations are in progress. See original paper for figures.