UNIQUENESS CLASSES FOR THE SOLUTION OF THE CAUCHY PROBLEM FOR LINEAR EQUATIONS WITH RAPIDLY GROWING COEFFICIENTS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.944

MATHEMATICS

Ya. I. Zhitomirskii

UNIQUENESS CLASSES FOR THE SOLUTION OF THE CAUCHY PROBLEM FOR LINEAR EQUATIONS WITH RAPIDLY GROWING COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 21 IV 1966)

Consider the Cauchy problem for the equation

$$\frac{\partial u(x,t)}{\partial t} = \sum_{k=0}^{n} q_k(x) \frac{\partial^k u(x,t)}{\partial x^k}, \tag{1}$$

$$-\infty < x < \infty, \quad t \geq 0, \quad n > 1,$$

with the initial condition $u(x,0) = \varphi(x)$.

In our preceding note (^{1}), classes of uniqueness and nonuniqueness of the solution of the Cauchy problem for (1) (and also for equations of a more general form) were indicated in the case when the coefficients of this equation are "slowly" growing functions as $|x| \to \infty$. In the present note we shall describe classes of uniqueness and nonuniqueness of the solution of the Cauchy problem for equation (1) under conditions on the coefficients $q_k(x)$ that allow their rapid growth as $|x| \to \infty$.

We shall assume:

a) $q_n(x) = a$, a complex constant;

b) $q_0(x)$ is a real, twice continuously differentiable function satisfying the conditions: $|q_0(x)| \to \infty$ as $|x| \to \infty$;

$$q_0''(x)(1 + |q_0(x)|^{1+1/n})^{-1} \in L_1(-\infty, \infty);$$

$$[q_0'(x)]^2(1 + |q_0(x)|)^{2+1/n})^{-1} \in L_1(-\infty, \infty);$$

c) the coefficients $q_k(x)$, $k = 1, \dots, n-1$, are complex-valued functions satisfying the conditions of "subordination" to the coefficient $q_0(x)$,

$$|q_k(x)|\left(1+|q_0(x)|^{(n-k-1)/n}\right)^{-1} \in L_1(-\infty,\infty),$$

$$k=1,\dots,n-1; \quad -\infty<x<\infty.$$

We note that condition b) does not restrict the order of growth of $q_0(x)$ as $|x|\to\infty$. When $q_0(x)$ grows rapidly, the coefficients $q_k(x)$, $k=1,\dots,n-2$, as condition c) shows, may also grow rapidly.

Let $\varphi=\arg a$, $-\pi<\varphi\le\pi$;

$$\mu_0=\begin{cases} \sin\dfrac{|\pi-2|\varphi||}{2n}, & \text{if } n=2m+1,\\ \sin\dfrac{|\varphi|}{n}, & \text{if } n=4m+2 \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}+\infty\\ & \text{and if } n=4m \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}-\infty,\\ \sin\dfrac{\pi-|\varphi|}{n}, & \text{if } n=4m \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}+\infty\\ & \text{and if } n=4m+2 \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}-\infty. \end{cases}$$

$$\mu_1=\begin{cases} \cos\dfrac{\varphi}{n}, & \text{if } n=2m+1 \text{ and } |\varphi|\le\frac{\pi}{2}\\ & \text{and if } n=2m \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}-\infty,\\ \cos\dfrac{\pi-|\varphi|}{n}, & \text{if } n=2m+1 \text{ and } |\varphi|\ge\frac{\pi}{2}\\ & \text{if } n=2m \text{ and } q_0(x)\xrightarrow[|x|\to\infty]{}+\infty. \end{cases}$$

Theorem 1. Let the coefficients of equation (1) satisfy conditions a), b), c), let $q_0(x)\to+\infty$ (or $q_0(x)\to-\infty$) as $|x|\to\infty$, and

$$\int_{-\infty}^{0}|q_0(x)|^{-1+1/n}\,dx<\infty, \qquad \int_{0}^{\infty}|q_0(x)|^{-1+1/n}\,dx<\infty. \tag{2}$$

Let $a(x)>0$ be a continuous even function such that $\inf a(x)=0$. Then the Cauchy problem for equation (1) has a unique solution in the class of functions $\{f(x,t)\}$

$$|D_x^j f(x,t)| \le C_f \frac{a(x)}{1+|q_0(x)|^{(n-1-2j)/2n}} \exp\left\{\mu_0\left|\int_0^x\left|\frac{q_0(t)}{a}\right|^{1/n}dt\right|+\beta t\right\},$$

$j = 0, 1, \dots, n-1$; $\beta > 0$ is an arbitrary (fixed) constant.

Theorem 2. Let the coefficients of equation (1) satisfy conditions a), b), c), (2), let $q_0'(x)$ not change sign for sufficiently large values of $|x|$; let $q_0(x) \to +\infty$ (or $q_0(x) \to -\infty$) as $|x| \to \infty$; and let the coefficients $q_k(x)$, $k = 1, \dots, n-1$, have continuous derivatives up to order k, with

$$|q_k^{(i)}(x)| \le A(1 + |q_0(x)|^{(n-k+i)/n}), \qquad i = 0, 1, \dots, k.$$

Then, in the class of functions $\{f(x,t)\}$

$$|D_x^j f(x,t)| \le C_f \exp\left\{ \mu \left| \int_0^x \left| \frac{q_0(t)}{a} \right|^{1/n} dt \right| + \beta t \right\}, \qquad j = 0, 1, \dots, n-1,$$

for some $\beta > 0$ and any $\mu > \mu_1$, uniqueness of the solution of the Cauchy problem for equation (1) does not hold.

Theorem 1 shows that, for $\mu_0 > 0$, the uniqueness classes for the solution of the Cauchy problem for equation (1) expand as the order of growth of $|q_0(x)|$ increases. If, however, $\mu_0 = 0$, then Theorem 1 guarantees uniqueness of the solution of the Cauchy problem for equation (1) only in a narrow class of functions, and one that is the narrower the greater the order of growth of $|q_0(x)|$ as $|x| \to \infty$.

Thus, for example, this class is certainly contained in L_2 (when $a(x) \to 0$). In a broader class of functions, uniqueness of the solution of the Cauchy problem for equation (1) when $\mu_0 = 0$, generally speaking, does not hold, as shown by the results of [2], where for second-order equations ($n = 2$; in this case $\mu_0 = \mu_1$) a more exact description was obtained both of the uniqueness classes for the solution of the Cauchy problem and of the nonuniqueness classes.

The following theorems describe uniqueness and nonuniqueness classes for the solution of the Cauchy problem for equation (1) in cases where the coefficients of this equation behave differently as $x \to +\infty$ and as $x \to -\infty$.

Theorem 3. Let $H(x) > 0$ be a function monotone for $x > 0$, satisfying the condition $\int^{\infty} [H(x)]^{1-n}\, dx = \infty$, and let $a(x) > 0$ be a continuous-

for $x < 0$ the function $\inf_{x<0} a(x) = 0$. Suppose that the coefficients $q_k(x)$, $k = 0, 1, \dots, n-1$, of equation (1) satisfy the following conditions:

I. For $x \le 0$:

1. $q_0(x)$ is a real twice continuously differentiable function; $|q_0(x)| \to \infty$ as $x \to -\infty$; the functions $q_0''(x)|q_0(x)|^{-1-1/n}$ and $[q_0'(x)]^2|q_0(x)|^{-2-1/n}$ belong, for some $x_0 \ge 0$, to the space $L_1(-\infty, -x_0)$.

2. The coefficients $q_k(x)$ are continuous functions, and

$$q_k(x)|q_0(x)|^{(k+1-n)/n} \in L_1(-\infty, -x_0), \quad k = 0, 1, \dots, n-1.$$

3.

$$\int_{-\infty}^{0} |q_0(x)|^{-1+1/n}\, dx < \infty.$$

II. For $x \geq 0$:

1. The coefficients $q_k(x)$ have continuous derivatives, and

$$\sup_{0 \leq t \leq x} |q_k'(t)| \leq C \frac{1 + |q_k(x)|}{1 + x},$$

$$|q_k(x)| \leq [h(x)]^{n-k}, \quad k = 0, 1, \dots, n-1; \quad h(x) > 0$$

is some function monotone for $x > 0$, satisfying the condition

$$\int^{\infty} [h(x)]^{1-n}\, dx = \infty.$$

2. $\operatorname{Re} a \neq 0$ (or otherwise one should assume that the roots $W_j(x, \tau)$, $j = 1, \dots, n$, of the equation

$$\sum_{k=0}^{n} q_k(x) W^k = \sigma_1 + i\tau, \quad \sigma_1 > 0,$$

are such that the differences $\operatorname{Re} W_l(x, \tau) - \operatorname{Re} W_k(x, \tau)$, for any l and k and all values of x, preserve their sign).

3. For $n = 2m + 1$, $\operatorname{Im} a \neq 0$.

Then the Cauchy problem for equation (1) has a unique solution in the class of functions $\{f(x, t)\}$

$$|D_x^j f(x, t)| \leq C_f G_j(x) \exp \beta t, \quad j = 0, 1, \dots, n-1,$$

for any (fixed) $\beta > 0$,

$$G_j(x) = \begin{cases} \exp \displaystyle\int_0^x H(t)\, dt, & \text{for } x \geq 0, \\ \dfrac{a(x)}{1 + |q_0(x)|^{(n-1-2j)/2n}} \exp \left[\mu_0 \displaystyle\int_x^0 \left| \frac{q_0(t)}{a} \right|^{1/n} dt \right], & \text{for } x \leq 0. \end{cases}$$

Theorem 4. Let $H(x) > 0$ be a function monotone for $x > 0$, satisfying the condition

$$\int^{\infty} [H(x)]^{1-n} dx < \infty,$$

and let $\mu > \mu_1$. Suppose that the coefficients of equation (1) satisfy, for $x \leq 0$, the conditions of Theorem 2, and for $x \geq 0$ the following conditions:

1. $q_k(x)$, $k = 0, 1, \dots, n-1$, have continuous derivatives up to order k, and

$$|q_k^{(r)}(x)| \leq [h(x)]^{n-k+r}, \quad k = 0, 1, \dots, n-1,$$

$$r = 0, 1, \dots, k,$$

where $h(x) > 0$ is some monotone function satisfying the conditions

$$\int^{\infty} [h(x)]^{1-n}\, dx = \infty, \qquad xh(x) = o(1) \int_0^x H(t)\, dt.$$

Then the uniqueness of the solution of the Cauchy problem for equation (1) in the class of functions

$$\left| D_x^j f(x,t) \right| \leq C_f F(x) \exp \beta t$$

for some $\beta > 0$ and

$$F(x) = \begin{cases} \exp \displaystyle\int_0^x H(t)\, dt, & \text{for } x > 0, \\ \exp \left[\mu \displaystyle\int_x^0 \left| \frac{q_0(t)}{a} \right|^{1/n} dt \right], & \text{for } x < 0 \end{cases}$$

does not hold.

The proofs of the indicated theorems are based on clarifying the question of the existence of a nontrivial solution $y(x, \lambda)$, analytic in some right half-plane and satisfying certain estimates there, of the ordinary differential equation

$$\sum_{k=0}^{n} q_k(x) y^{(k)}(x, \lambda) = \lambda y(x, \lambda). \tag{3}$$

Here the most essential point is the study of the asymptotics of solutions of equation (3) on vertical lines ($\lambda = \sigma_1 + i\tau,\ \sigma_1 > 0$) for all values of x, $-\infty <$

$x < \infty$, and also in the right half-plane ($\operatorname{Re}\lambda \geq \sigma_0$) for sufficiently large values of $|x|$. We note here the following two theorems.

Theorem 5. Suppose conditions a), b), c) are satisfied. Then equation (3) has, for $\lambda = \sigma_1 + i\tau$ (for sufficiently large $\sigma_1 > 0$) and $\tau \geq \tau_0 > 0$ (or $\tau \leq -\tau_0$), n linearly independent solutions $y_j(x, \tau)$, $j = 0, 1, \dots, n-1$, such that

$$D_x^k y_j(x,\tau) = \left(\frac{\lambda - q_0(x)}{a}\right)^{k/n-(n-1)/2n} \left[\exp\frac{2\pi ijk}{n} + \delta_{jk}(x,\lambda)\right] \exp\int_0^x W_j(t,\lambda)\,dt, \tag{4}$$

where

$$W_j(x,\lambda) = \left(\frac{\lambda - q_0(x)}{a}\right)^{1/n} \exp\frac{2\pi j}{n}; \quad \delta_{jk}(x,\lambda) \to 0 \quad \text{as } |\tau| \to \infty$$

uniformly with respect to x, $-\infty < x < \infty$, $k = 0, 1, \dots, n-1$.

Theorem 6. Suppose the coefficients of equation (3) satisfy the conditions of the preceding theorem and $q_0'(x)$ does not change sign for sufficiently large values of $|x|$. Suppose the functions $p(t)$ and $q(t)$ are defined by the relations

$$|q_0(p(t))| = t, \quad p(t) > 0; \qquad |q_0(q(t))| = t, \quad q(t) < 0.$$

Then there exists a constant $k > 0$ such that equation (3) has n linearly independent solutions $y_j(x, \lambda)$, $j = 0, 1, \dots, n-1$, for which, for $x \geq p(k|\lambda|)$ and for $x \leq q(k|\lambda|)$, the asymptotic formula (4) holds, with $\delta_{jk}(x, \lambda) \to 0$ (as $\operatorname{Re}\lambda \to \infty$) uniformly with respect to x in the regions $x \geq p(k|\lambda|)$ and $x \leq q(k|\lambda|)$.

Kharkov Automobile and Highway Institute

Received
20 IV 1966

REFERENCES

1. Ya. I. Zhitomirskii, DAN, 172, No. 6 (1967).

2. Ya. I. Zhitomirskii, DAN, 171, No. 1 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.