The synthesis of optimal controls in nonlinear oscillatory systems of second order

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Date: 1967-01-01T00:00:00+00:00

Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS, 1967, VOL. III, NO. 3

Synthesis of Optimal Controls in Nonlinear Second-Order Oscillatory Systems

Consider a controlled object whose motion is described by a second-order differential equation with a one-dimensional control domain:

$$-1 \leq u \leq 1.$$

The introduction of variables reduces the equation to a normal second-order system:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = f(x_1, x_2, u).$$

We assume that the function f is continuous and continuously differentiable with respect to x_1, x_2, u, and twice continuously differentiable with respect to x_1, x_2. Furthermore, it satisfies the following conditions:

$$f(0,0,1) > 0, \quad f(0,0,-1) < 0, \quad \frac{\partial f(x_1,x_2,u)}{\partial u} > 0 \text{ for all } x_1, x_2, u;$$

$$\frac{\partial f(x_1,x_2,u)}{\partial x_1} < 0 \text{ for all } x_1, x_2, u;$$

$$\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} < 0, \quad \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \leq 0$$

for $u = \pm 1$ and all x_1, x_2. Condition (7) implies that for $u = \pm 1$, the quadratic form takes only non-positive values. These conditions are satisfied, for example, by the linear object $\ddot{x} + x = u$ (see [?], pp. 34-43), as well as by nonlinear objects differing from it by a "small" convex addition. Specifically, let $\phi(x_1, x_2, u)$ be an arbitrary function convex in x_1, x_2 (i.e., satisfying condition (7)) with bounded first derivatives $\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}$. Then, setting

$$f(x_1, x_2, u) = -x_1 + u + \mu\phi(x_1, x_2, u),$$

we obtain for sufficiently small μ an object (3) satisfying all conditions (4)-(7). Additionally, if f takes the form $f(x_1, x_2, u) = -x_1 + g(x_2, u)$, where g satisfies $g(0, 1) > 0, g(0, -1) < 0, \frac{\partial g}{\partial u} > 0, \frac{\partial^2 g}{\partial x_2^2} \leq 0$, then all conditions (4)-(7) are also fulfilled (see [?], p. 510).

For the object described by relations (2)-(7), we consider the problem of time-optimal arrival at the origin. As we shall see, the synthesis of optimal controls for this nonlinear object is qualitatively the same as for the linear object $\ddot{x} + x = u$; that is, the optimal trajectories approach the origin as spirals (see Fig. 13, p. 41 of [?] and Fig. 10 below). Interestingly, this "oscillatory" character of the optimal trajectories is linked to condition (6), which can be termed the condition of "strong negativity" of the derivative $\frac{\partial f}{\partial x_1}$. As shown in [?], replacing this condition with the condition of non-negativity of the derivative leads to each optimal control having only one switching, and the synthesis pattern resembles that of the linear object $\ddot{x} = u$ (see [?], pp. 29-34, particularly Fig. 7).

We now proceed to solve the time-optimal problem for object (2)-(7). First, we find all phase trajectories satisfying the maximum principle ([?], p. 26). The Hamiltonian H for this object is:

$$H = \psi_1 x_2 + \psi_2 f(x_1, x_2, u).$$

The system of equations for the adjoint variables is:

$$\dot{\psi}_1 = -\psi_2 \frac{\partial f}{\partial x_1},$$
$$\dot{\psi}_2 = -\psi_1 - \psi_2 \frac{\partial f}{\partial x_2}.$$

The maximum condition (along an optimal trajectory, H reaches its maximum with respect to u) implies that the expression $\psi_2 f(x_1, x_2, u)$ reaches its maximum. Since $\frac{\partial f}{\partial u} > 0$, the function f is monotonically increasing in u, thus:

$$u = \text{sign } \psi_2. \quad (10)$$

To understand the behavior of ψ_2, we examine the law of change for the angle $\alpha = \arctan(\psi_1/\psi_2)$. According to (9), we have:

$$\frac{d}{dt} \arctan \frac{\psi_1}{\psi_2} = \frac{\dot{\psi}_1 \psi_2 - \psi_1 \dot{\psi}_2}{\psi_1^2 + \psi_2^2} = \frac{\psi_1^2 + \psi_1 \psi_2 \frac{\partial f}{\partial x_2} - \psi_2^2 \frac{\partial f}{\partial x_1}}{\psi_1^2 + \psi_2^2}.$$

The discriminant of the quadratic form in the numerator is $(\frac{\partial f}{\partial x_2})^2 + 4\frac{\partial f}{\partial x_1}$. According to (6), this discriminant is negative, which is precisely why condition (6) was formulated. Thus, the numerator maintains a constant sign. It is clear that it remains positive (setting $\psi_2 = 0$ yields $\psi_1^2 > 0$), meaning $\arctan(\psi_1/\psi_2)$ monotonically increases, or equivalently, the vector $\{\psi_1(t), \psi_2(t)\}$ rotates clockwise.

Furthermore, for $u = \text{const}$, we have:

$$\frac{d}{dt}\arctan\frac{x_2}{x_1} = \frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2 + x_2^2} = \frac{x_1 f(x_1, x_2, u) - x_2^2}{x_1^2 + x_2^2}.$$

According to (7), the quadratic form in the numerator maintains a negative sign for $u = \text{const}$. In other words, on each segment of the solution where $u = \text{const}$, the phase velocity vector $\{x_1(t), x_2(t)\}$ rotates clockwise.

Let x_0 be an arbitrary point in the phase plane where $x_0 \neq 0$. Consider the solution $x(t), \psi(t)$ of the system (3), (9) for $u = 1$, satisfying the initial conditions:

$$x_1(0) = x_{1,0}, \quad x_2(0) = x_{2,0}, \quad \psi_1(0) = 1, \quad \psi_2(0) = 0. \quad (13)$$

From (9) and (13), we find $\dot{\psi}_2(0) = -1$. Consequently, $\psi_2(t) > 0$ for small negative t. Let $\tau_+(x_0)$ be the negative root of $\psi_2(t)$ closest to zero. On the interval $\tau_+ < t < 0$, the function ψ_2 remains positive. We denote the segment of the trajectory $x(t)$ corresponding to this time interval as $K_+(x_0)$. This segment ends at x_0; its starting point is denoted by $\xi_+(x_0)$. Along this segment, $\psi_2(t) > 0$ and $u = 1$, satisfying the maximum principle. Since $f(x_1, x_2, 1) \neq 0$, the point x_0 is not an equilibrium, and the phase velocity vector is non-zero along the entire arc.

Similarly, using initial conditions $\psi_1(0) = -1, \psi_2(0) = 0$, we find a solution for $u = -1$. Let $\tau_-(x_0)$ be the negative root of $\psi_2(t)$ closest to zero. The corresponding trajectory segment is $K_-(x_0)$, with the starting point denoted by $\xi_-(x_0)$. This segment also satisfies the maximum principle.

We now prove that if x_0 lies on the x_1-axis, then both points $\xi_+(x_0)$ and $\xi_-(x_0)$ also lie on the x_1-axis. In this case, $K_+(x_0)$ is a convex arc located entirely on one side of the x_1-axis, with tangents at its ends parallel to the x_2-axis (Fig. 1a). If x_0 does not lie on the x_1-axis, then $\xi_+(x_0)$ and $\xi_-(x_0)$ lie on opposite sides of the axis, and the arc intersects the x_1-axis exactly once (Fig. 1b). This follows from the fact that the vector $\psi(t)$ rotates clockwise, and at two consecutive zeros of $\psi_2(t)$, the values of $\psi_1(t)$ must have opposite signs.

The convexity of these arcs follows from the unidirectional rotation of the tangent vector. The fact that the arc $K_+(x_0)$ can intersect the x_1-axis at most once is proven by contradiction: if it intersected twice, the phase velocity vector would be parallel to the x_1-axis at two points, which would contradict the property that $\psi_2(t)$ has no zeros between τ_+ and 0.

Next, we construct the trajectories arriving at the origin that satisfy the maximum principle. Starting from the origin o, we construct the arc $K_+(o)$. This arc lies entirely below the x_1-axis, and its starting point ζ_1 lies on the x_1-axis (Fig. 4). For any point z on $K_+(o)$, we can define an arc $K_-(z)$ ending at z. Let the starting point of this arc be ζ_2. Continuing this process, we construct a sequence of arcs $K_+(o), K_-(\zeta_1), K_+(\zeta_2), ...$, forming a trajectory $\eta_+(\zeta_0)$. Similarly, we construct $\eta_-(\zeta_1)$ starting with $K_-(o)$. All such trajectories satisfy the maximum principle and consist of alternating segments where $u = +1$ and $u = -1$.

The topological structure of these trajectories in the phase plane forms a series of curvilinear quadrilaterals Q_i^-, Q_i^+. Each trajectory, except those forming the boundaries of these quadrilaterals, enters a quadrilateral once and exits it at a non-zero angle. The switching line consists of the union of the arcs $K_+(o), K_-(o)$ and their subsequent mappings.

Conclusion

The final result for the time-optimal control of the object (2)-(7) is as follows: the optimal trajectories arriving at the origin are spirals consisting of alternating segments corresponding to $u = +1$ and $u = -1$. The switching line consists of the constructed arcs, and the synthesis of optimal controls is realized by a function $u(x_1, x_2)$ that takes the value -1 above the switching line and $+1$ below it. The qualitative behavior of the optimal trajectories is shown in Fig. 10.

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Received by the Editorial Board September 1966. V.A. Steklov Mathematical Institute.

Figures

DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL. III, NO. 12

UDC 517.934:62.50

REGULARIZATION OF ONE PROBLEM OF PURSUIT

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In this paper, we discuss the specifics of the pursuit problem [1—6] for objects of the same type in the case where the goal of the pursuit is a meeting not in all [3], but only in part of the phase coordinates. A proof of the results announced in the note [4] is given.

§ 1. Let us assume that the change in the phase vectors $y(t) = \{y_i(t)\}$ and $z(t) = \{z_i(t)\}$ $(i = 1, \ldots, n)$ — respectively, of the pursuing and pursued objects — is determined in time by systems of linear differential equations

$$\dot{y} = Ay + Bu, \tag{1.1}$$

$$\dot{z} = Az + Bv, \tag{1.2}$$

where $u = \{u_j\}$ and $v = \{v_j\}$ $(j = 1, \ldots, r < n)$ are control vectors, constrained by integral conditions of the form (see [3], p. 209)

$$\int_{\tau}^{\infty} \|u(t)\|^2 \, dt < \mu^2(\tau), \quad \int_{\tau}^{\infty} \|v(t)\|^2 \, dt < \nu^2(\tau), \tag{1.3}$$

A and B — are constant matrices of the corresponding dimensions.

Let there be selected certain phase coordinates y_{ik} and z_{ik} $(k = 1, \ldots, m < n)$, the coincidence of which at the moment of meeting $t = \vartheta$ constitutes the goal of the pursuit. Without loss of generality, we can consider that these selected coordinates are the first m coordinates of the phase vectors y and z. It is convenient to consider, the sets of coordinates $\{y_i\} = y_{[m]}$ and $\{z_i\} = z_{[m]}$ $(i = 1, m)$ as vectors $q = \{q_i\}$ $(i = 1, \ldots, m)$ in the m-dimensional space Q.

Let us approach the problem of achieving a meeting of opposing objects as a differential positional game of two persons with perfect information [1—6]. In such a game, at each given moment of time $t = \tau$, all phase coordinates $y_i(\tau)$, $z_i(\tau)$ $(i = 1, \ldots, n)$ are known to each partner, as well as the estimates $\mu(\tau)$ and $\nu(\tau)$ of the control resources rettaining for the time $t \geq \tau$ (1.3). However, information about the current and future choice of $v(t)$ $(t \geq \tau)$ is absest. Moroever, the payment of the game is the time $T_{u,v} = \vartheta_{u,v} - \tau$ to the meeting of the motions $y(t)$ (1.1) and $z(t)$ (1.2) in part of the selected coordinates. Consequently, the first player (pursuer) seeks to minimize, and the stpond player (pursued) seeks to maximize the maxinmize the specified quality index of the game. By the nature of the positional game, the control $u(t)$ at each moment of time $t = \tau$ most naturally should be formed according to the principle of feedback based on the measurement of the values $y(t)$, $z(t)$, $\mu(\tau)$ and $\nu(\tau)$, i. e.

Figure 1: Figure 1

$$u(\tau) = u\,[y(\tau),\ z(\tau),\ \mu(\tau),\ \nu(\tau)]. \qquad (1.4)$$

For the control v, it is possible to use both program control $v(t)$ and control based on the principle of feedback, i.e., in the form

$$v(\tau) = v\,[y(\tau),\ z(\tau),\ \mu(\tau),\ \nu(\tau)]. \qquad (1.5)$$

Strategies u and v, i.e., the set of functions of the form (1.4) and (1.5), will be considered admissible if during their realization

$$u[t] = u\,[y(t),\ z(t),\ \mu(t),\ \nu(t)],$$

$$v[t] = v\,[y(t),\ z(t),\ \mu(t),\ \nu(t)] \quad \text{or} \quad v = v(t)$$

the restrictive conditions (1.3) are not violated and equations (1.1), (1.2) retain meaning (possibly — generalized). Summarizing the above, let us pose the following problem.

Problem 1. Among the admissible strategies u in (1.4) and v in (1.5), find such optimal strategies u^0 and v^0, that for any initial data $y(t_0)$, $z(t_0)$, $\mu(t_0)$, $\nu(t_0)$ (from a given region of their possible change) the condition holds

$$T_{u^0,v^0} = \min_u \max_v T_{u,v}. \qquad (1.6)$$

It should be emphasized that the differential game under consideration, due to the integral nature of constraints (1.3) does not, generally speaking, have a saddle point (see [3], example 7.1), therefore in problem 1 it is required to provide only condition (1.6), i.e. it is required to solve the pursuit problem only from the point of view of the interests of the pursuer. Formulated problem 1 is solved using areas of reachability [5] in work [3] in the case when $m = n$, i.e. when it is required to realize the meeting in all phase coordinates. Let us try to solve the problem in the same way here, when $m < n$. For this purpose, we select some number $\theta > \tau$ and construct areas of reachability $G^{(1)}\,[y\,(t),\ \mu\,(t),\ \theta]$ and $G^{(2)}\,[z\,(t),\ \nu\,(t),\ \theta]$, i.e. such largest areas in the m-dimensional space Q, for each point of which $y'_{[m]}$ and $z'_{[m]}$ it is possible to build program controls $u'(t)$ and $v'(t)$, degree constraints...

$$\int_\tau^s \|u'(t)\|^2\,dt \leqslant \mu^2(\tau), \quad \int_\tau^s \|v'(t)\|^2\,dt \leqslant \nu^2(\tau) \qquad (1.7)$$

and transferring objects (1.1) and (1.2) over time $T = \theta - \tau$ from positions $y\,(\tau)$ and $z\,(\tau)$ to positions $y_{[m]}\,(\theta) = y'_{[m]}$ and $z_{[m]}\,(\theta) = z'_{[m]}$ respectively. Assume that system (1.1), and consequently system (1.2), are fully controllable in terms of the selected coordinates on any interval $[\tau, \theta]$, which can occur in the problem. This condition is satisfied in any case, if the rank of the matrix $K = [B,\ AB,\ \ldots,\ A^{n-1)}B]$ is greater than or equal to m and space Q lies in the subspace $K_{[s]}$ (τ = tang $K > m$), fornnd formed by the linearly independent columns of this matrix.

As is known from the theory of optimal control, to obtain, for example, the area of reachability $G^{(1)}\,[y\,(t),\ \mu\,(t),\ \theta]$, it is sufficient to choose control $u\,(t)$ ($\tau < t < \theta$) in the form

$$u\,(t) = H^{\text{im1}}\,(\theta - t)\,l, \qquad (1.8)$$

where H^{im1} — matrix composed of the first m rows of the impulse transition matrix $H\,(\theta - t) = X\,(\theta - t) \cdot B$; $X\,(\theta - t)$ — the fundamental matrix

Figure 2: Figure 2

(1.8), (1.10), we finally oblyin the relation, defineing the objects $G^{(1)}$ [$y(\tau)$, $\mu(\tau)$, θ] in the prostpace Q

$$y_{[m]}(\theta) = X^{[m]}(\theta - \tau)\, y(\tau) + \int_{\tau}^{\theta} H^{[m]}(\theta - t)\, H^{[m]\prime}(\theta - t)\, l dt, \qquad (1.9)$$

where $X^{[m]}$ is a matrix composed of the first m rows of the fundamental matrix : X. From (1.9) we man find

$$l = D^{-1}\varphi_{[m]}. \qquad (1.10)$$

There D^{-1} is the matrix ibparse to the matrix

$$D = \int_{\tau}^{\theta} H^{[m]}(\theta - t)\, H^{[m]\prime}(\theta - t)\, dt, \qquad (1.11)$$

$$\varphi_{[m]} = y_{[m]}(\theta) - X^{[m]}(\theta - \tau)\, y(\tau), \qquad (1.12)$$

and the matrix D^{-1} necessarily exists vcuests, eecn tonko system (1.1) is fully ynerabliable with respect to a part of thein coordinates. Taking into acvoint further (1.7), (1.8), (1.10), we finally oblyin the relation, defineing the oblects $G^{(1)}$ [$y(\tau)$, $\mu(\tau)$, ϑ] in the prostpace Q

$$G_1(q_1, \ldots, q_m) \equiv (D^{-1}\varphi_{[m]}, \varphi_{[m]}) \leqslant \mu^2. \qquad (1.13)$$

In a sodfher way, it is possible to costrout the oblects $G^{(2)}$ [$z(\tau)$, $\nu(\tau)$, ϑ]. In the kond, we will onlyin

$$G_2(q_1, \ldots, q_m) \equiv (D^{-1}\psi_{[m]}, \psi_{[m]}) \leqslant \nu^2, \qquad (1.14)$$

whe $\psi_{[m]} = z_{[m]}(\theta) - X^{[m]}(\theta - \tau)\, z(\tau)$.

Lyct then $\theta = \theta_0$ be the moment aurornation of process $z(t)$ by the process $y(t)$, i.e. taked moment $t = \theta$, larga the first time the oblects $G^{(2)}$ [$z(\tau)$, $\nu(\tau)$, ϑ] fade is enllernly inside the oblactn $G^{(1)}$ [$y(\tau)$, $\mu(\tau)$, ϑ]. It is obvious, un with contruuous axmenage of θ the organivened sanyted and convyxex oblacts $G^{(1)}$ and $G^{(2)}$ are contruiously deformised and therefor it turns out, tho at a moment $\theta = \theta_0$ ux granunits toopracates at hrast at one tovke. The funktion G_1 (1.13) and G_2 (1.14) turned to beat kuadpratic fynktions of the se tine of the varieables $q_1, \ldots, q_m$, and, therefore, the releations (1.13) and (1.14) will nns define in the prostpractice Q sodofin and identicallly orientely orented ellinsouds. From see there areosome coillowing circernance directly follows: if the oblacth $G^{(3)}$ is co-non codepwined a oblacts $G^{(1)}$ u granunis ux contact contait, to this contact may nakes sμlace, alibe a simner single point q^0, or there will be bekonevly many points of contact and thern the ukasased objects. spocty connadate.

Let us asusume, that take snvenue $y(\tau)$, $z(\tau)$, $\mu(\tau)$, $z(\tau)$, $\mu(\tau)$, $\nu(\tau)$ and kotopich number ϑ cymect yet is oblactn $G^{(2)}$ [$z(\tau)$, $\nu(\tau)$, θ_0] touches us the oblect. $G^{(1)}$ [$y(\tau)$, $\mu(\tau)$, ϑ_0] as touching to a single point q^0. Tho entermine this point, it is soctatiovt [5] to selve cillywing problem for a conditional minimum:

$$\min_q G_2(q_1, \ldots, q_m) = \nu^2 \qquad (1.15)$$

object to $G_1(q_1, \ldots, q_m) = \mu^2$. Solving the sadraw (1.15) by the obuunl method of Lagrangers, we find

$$q^0 = \frac{1}{\mu - \nu}\, X^{[m]}(\theta_0 - \tau)\, [\mu z - \nu y]. \qquad (1.16)$$

Figure 3: Figure 3

At the same time, we obtain the final equation for determining the moment of absorption

$$(D^{-1}\, c_{[m]},\, c_{[m]}) - (\mu - \nu)^2 = 0, \tag{1.17}$$

where

$$c_{[m]} = -\, X^{[m]}\,(\vartheta - \tau)\, x(\tau),\ x(\tau) = y(\tau) - z(\tau). \tag{1.18}$$

The moment of absorption will be the smallest positive root $\vartheta = \vartheta_0$ of equation (1.17). Let us now construct the controls $u_0[t]$ and $v_0[t]$, aiming (see [5], p. 7) the movements $y(t)$ and $z(t)$ at each moment of time $t = \tau$ to the point $y_{[m]}(\vartheta_0) = z_{[m]}(\vartheta_0) = q^0$ (1.16). In what follows, these controls are called extremal, and the rule for aiming movements to point q^0 q^0 is the rule of extremal aiming. Taking into account (1.8), (1.10), (1.12), (1.16), we obtain

$$u_0[t] = u_0\,[y(t),\ z(t),\ \mu(t),\ \nu(t)] = \frac{\mu}{\mu - \nu}\, w^0[t], \tag{1.19}$$

where

$$w^0[t] = H^{[m]\prime} D^{-1} c_{[m]}. \tag{1.20}$$

Analogously, we find

$$v_0[t] = v_0\,[y(t),\ z(t),\ \mu(t),\ \nu(t)] = \frac{\nu}{\mu - \nu}\, w^0[t]. \tag{1.21}$$

By direct calculation, it can be established that $w^0[t]$ (1.20) is the solution to the problem of transferring the system

$$\dot{x} = Ax + Bw \tag{1.22}$$

from state $x = x(\tau)$ to position $y_{[m]}(\vartheta^0) - z_{[m]}(\vartheta^0) = x_{[m]}(\vartheta^0) = 0$ under the constraint

$$\Big[\int_{\tau}^{\infty} \|w[t]\|^2\, dt\Big]^{1/2} \le \zeta(\tau) = \mu(\tau) - \nu(\tau) \tag{1.23}$$

and under the condition

$$T^0 = \vartheta^0 - \tau = \min_{w} T. \tag{1.24}$$

In particular, it turns out that the moment of absorption ϑ_0 determined from (1.17) coiniades with the moment ϑ^0 of arrival of the movement $x(t)$ at position $x_{[m]} = 0$.

In work [3], where the discussion was about meeting in all phase coordinates ($m = n$), it was established that the extremal controls u_0 (1.19) and v_0 (1.21) are optimal strategies that solve problem 1 on pursuit. This fact took place because with $u = u_0$ (1.19) and for any admyssible v all the time before the meeting, a situation could not not arise where the boundaries of the reachability regions $G^{(1)}$ (1.13) and $G^{(2)}$ (1.14) such the reachability regions $G^{(1)}$ (1.13) and $G^{(2)}$ (1.14) touch at more than one point, if only at the initial moment of pursuit $t = t_0$ the indicated contact occurred at a single point $q^0(t_0)$ (1.16). The matter is more complicated in the case considered $m < n$. As follows from what follows, here the extremal control u_0 (1.19) no longer guarantees meeting for any admissible control v of movements $y(t)$ and $z(t)$ during the time $\tau \le t \le t \le \tau + T^0(\tau)$, and rule of extreemal extremal aiming does not provide $\min_u \max_v T = T^0$. In other words, the extremal controls $u = u_0$ (1.19) and $v = v_0$ (1.21) do not constitute a pair of parssi-optimal strategies for $m < n$. This assertion is proved by the following example.

Figure 4: Figure 4

§ 2. Let the systems (1.1) and (1.2) have the form

$$\dot{y}_1 = y_3,\ \dot{y}_3 = u_1,\ \dot{y}_2 = y_4,\ \dot{y}_4 = u_2, \tag{2.1}$$

$$\dot{z}_1 = z_3,\ \dot{z}_3 = v_1,\ \dot{z}_2 = z_4,\ \dot{z}_4 = v_2, \tag{2.2}$$

and it is required to carry out the meeting only along coordinates y_1, y_2 and z_1, z_2. The extremal control u_0 in view of (1.19), (1.20) has the form

$$u_0 = \left\{ -\frac{3}{|T^0|^2}\,\frac{\mu}{\zeta}\,(x_1 + T^0 x_3);\ -\frac{3}{|T^0|^2}\,\frac{\mu}{\zeta}\,(x_2 + T^0 x_4) \right\}, \tag{2.3}$$

where $\zeta = \mu - \nu$, and the quantiny T^0 is the samilest positive root of the equation (1.17)

$$\zeta^2 T^3 - 3(x_1 + x_3 T^0)^2 - 3(x_2 + x_4 T^0)^2 = 0. \tag{2.4}$$

Moreover, that at the initial moment of pursuit $t = t_0 = 0$ the position took positoce

$$z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0,$$
$$y_2(0) = y_4(0) = 0,\ y_1(0) = y_{10},\ y_3(0) = y_{30}, \tag{2.5}$$

and, in addition, assume, that the pursulyed while chose for some time $t_0 < t \leqslant t^* < \vartheta$ the control $v(t) = \{v_1(t);\ v_2(t)\} \equiv 0$. Then for all the time, while $v(t) \equiv 0$, the equalities will be fulfillled $z_1(t) = z_2(t) = z_3(t) = z_4(t) = y_2(t) = y_4(t) \equiv 0$, $v(t) \equiv v(0) = v_0$, and the process of pursuit, being led by the pursuer according to the rule of extremal aiming (2.3), will be sama-described by the system of differential equations

$$\begin{aligned}
\dot{y}_1 &= y_3,\\
\dot{y}_3 &= -\sqrt{3}\ \frac{\mu}{\sqrt{T^0}}\ \operatorname{sgn}(y_1 + y_3 T^0),\\
\dot{\mu} &= -\frac{3\mu}{2T^0},\\
\dot{T}^0 &= -1 - \frac{v_0}{(\mu - v_0) - \dfrac{2}{\sqrt{3}}\, y_3\, \dfrac{1}{\sqrt{T^0}}\ \operatorname{sgn}(y_1 + y_3 T^0)}.
\end{aligned} \tag{2.6}$$

The lasest differential eppramation in the system (2.6) is obtained by formal calculation of the derisvative dT^0/dt implicitly from the ypuanion (2.4). Let us now try to indicate such initial condicitions

$$y_1(0) = y_{10},\ y_3(0) = y_{30},\ \mu(0) = \mu_0 > v_0,\ T^0(0) = T_0, \tag{2.7}$$

for retopich at the moment of time $t = t_*$ by dirtue of differential equastions (2.6) it turns out $\mu(t^*) = v(t^*) = v_0$. In such a clyae the reachability rüjects $G^{(1)}[y(t_*),\ \mu(t_*),\ \vartheta_0(t_*)]$ (1.13) and $G^{(2)}[z(t_*),\ v(t^*),\ \vartheta_0(t_*)]$ (1.14), being for ooher epummple crycles of radijs of $\{|T^0|^3 \mu^2/3\}$ and $\{|T^0|^3 v^2/3\}$ resspectv-rely, will turn out to coinidate.

Assume, that the decired initial condititions (2.7) exists and the moment $t = t_{*}$, when $\mu(t^*) = v(t^*)$, has arrived. Thun from (2.4) follows the parencity

$$\lambda(t^*) = y_1(t^*) + y_3(t^*)\, T^0(t^*) = 0, \tag{2.8}$$

Figure 5: Figure 5

indicating the coincidence of the centers of the above-mentioned circles. Assuming $v_0 = v(t^*) = 1$, $y_1(t^*) = -0.25$, $T^0(t^*) = 0.25$, we obtain from (2.8) $y_3(t^*) = 1$.

$$y_1(t^*) = -0.25,\ y_3(t^*) = 1,\ \mu(t^*) = 1,\ T^0(t^*) = 0.25 \quad (2.9)$$

Let us now take the values and we will, by interpreting system (2.6) backward, assuming, consequently, $t = -t' + f_9$. If in this case it turns out that for the initial data (2.9) during backward time solution of system (2.6) exists, then, obviously, as the required initial data (2.7) we can take any values of the resulting solution for $t' > f_0 = f_9$. It is not difficult to establish that the solution of the reversed system (2.6) for initial conditions (2.9) exists and is unique at least for $t' > f_0$, sufficiently large, and for $t_0 = f_9$, if only we determine the sign of the quantity $\lambda(t)$ for $t = f_0$, setting $\operatorname{sgn} \lambda(f_9) = -1$. It remains still to verify that the quantity $T^0(t')$ for $t' > f_9$, determined by the last equation of system (2.6), is the smallest positive root of the equation

$$F(t', T) = \xi^2(t')T^3 - 3\,|y_1(t') + y_3(t')T|^2 = 0. \quad (2.10)$$

This is confirmed by a direct numerical experiment. If system (2.6) is integrated in backward time with initial data (2.9), starting from the moment $t_0{}' = 0$ and, for example, up to the moment $t_1' = 0.16$, then as a result of the calculation we obtain

$$y_1(t') = -0.358510,\ y_3(t') = 0.338888,$$
$$\mu(t') = 1.946702,\ T^0(t') = 0.500000. \quad (2.11)$$

Simultaneously, it turns out that the resulting solution $T^0(t')$ at each moment of time $0 < t' < 0.16$ is the smallest positive root of equation (2.10). In Fig. 1 the deformation of the curve $F = F(t', T)$ is represented for the change of t' on the segment $[f_0, f_1]$.

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Thus, as the required initial data (2.7) we can take, for example, the values (2.11)

$$y_{10} = y_1(t'),\ y_{30} = y_3(t'),\ \mu_0 = \mu(t'),\ T_0 = T^0(t').$$

Integrating system (2.6) with these initial conditions, at the moment $t = -t_9 = 0.16$ we obtain, in this way, $\mu(t_9) = v(t_9)$, and, consequently, the attainability regions $G^{(1)}$ and $G^{(3)}$ turn out at this moment of time to be merged. But at such moments of time, when the attainability regions touch more than at one point, the rule of extremal aiming becomes, obviously, inefficient. Moreover, at such moments of time it is impossible to point out a strategy $u(1.9)$, which would keep the region $G^{(2)}$ inside the region $G^{(1)}$, if only the pursuer is not informed about the choice by the evader at these moments of time of the control $v(t)$. Consequently, the extremal control u_0 (1.19) for $m < n$, generally speaking, does not guarantee the meeting of motions at the moment of time $t < \phi_0$.

§ 3. Assume now, that there exists an admissible control of the form $u(t) = u^*[y(t), z(t), \mu(t), v(t)]$, which ensures the meeting of motions for a part of the selected coordinates at the moment $t < \tau + T^0 \cdot T^0(t)$ for any admissible behavior of the evader. In doing so, in accordance with the game formulation of the problem ! we must take into account that the pursuer is not guaranteed guaranteed against the choice of the evader's control.

Figure 6: Figure 6

$$v = v^* = \frac{\nu}{\mu}\ u^*. \quad (3.1)$$

But then in the process of movement when $t \geqslant \tau$, the equality will always be sa- satisfied

$$\nu(t)/\mu(t) = \nu_0/\mu_0 = \text{const}, \quad (3.2)$$

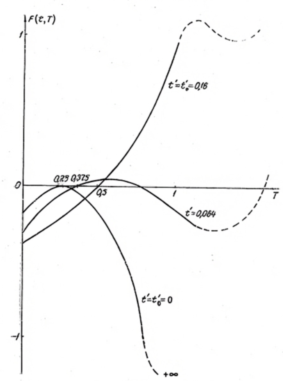

Fig. 1

which is obtained as a consequence of integrating the equations

$$\dot\mu = -\|u\|^2/2\mu,\ \dot\nu = -\|v\|^2/2\nu, \quad (3.3)$$

arising from the restrictive conditions (1.3). From the equality (3.2) it is deni- deluced that the control $\nu = \nu^*$ (3.1) is also admissible and, moreover, that on the segment $[\tau, \tau + T^0(\tau)]$ the inequality will hold

$$\iint_{\tau}^{\vartheta_0(\tau)} \|w^*|t|\|^2\, dt\Big]^{1/2} \leqslant \zeta(\tau) = \mu(\tau) - \nu(\tau), \quad (3.4)$$

where $w^* = u^* - v^*$. From the inequality (3.4) it follows that meeting is impossible when $t < \tau + T^0(\tau)$, поскольку $T^0 = \min_T T^0$ is the solution of the formulated

Figure 7: Figure 7

In § 1 the problem of optimal time-optimal (cm. (1.22), (1.23), (1.244). The meeting would have occurred at moment $t = t + T^0(\tau)$ only if $\omega^{\circledast}[r]$ coincided with $\omega^{\circledast}[r]$ (1.20), but then the control $u^k[r]$ would be extremal, an extremal control, as follows from the example considered in the previous paragraph, ensures a meeting at moment
$t < \tau + T^0(\tau)$.

Thus, we come to the conclusion that it is impossible in general to contruct a strategy of the form $u(\tau) = u[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$, which would guarantee a meeting of the motions in terms of of the phase coordinates at a moment $t \leq \tau + T^0(\tau)$ for any admissible behavior of the evader. Therefore, problem 1 on pursuit for $m < n$ becomes ill-posed and, consequently, the necessity for its regularization arises.

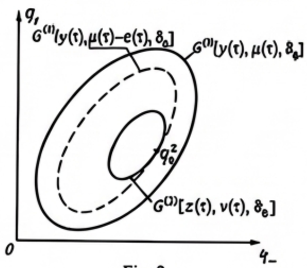

Fig. 2

§ 4. Let us dwell here on one method of regularization of the pursuit problem. In doing so, we will be guided by the considerations expressed in paper [6] (see § 5). Let us assume that the control resource of the pursuer is slightly increased. For convenience of notation, let us assume that $\mu(\tau)$ is already the resource increased by a small number $\varepsilon(\tau) > 0$. Let us construct the attainability domains $G^{(1)}[y(\tau), \mu(\tau) - \varepsilon(\tau), \theta_e]$ and $G^{(2)}[z(\tau), \nu(\tau), \theta_c]$ (see Fig. 2). Since θ_e is the moment of absorption of the process $z(t)$ by the process $y(t)$, under the condition that the pursuer possesses the resource $= y(\tau) - \varepsilon(\tau)$. Obviously, the domain $G^{(2)}[z(\tau), \nu(\tau), \theta_e]$ will lie strictly inside the domain $G^{(1)}[y(\tau), \mu(\tau), \theta_e]$, touching the boundaries of the domain $G^{(1)}[y(\tau), \mu(\tau) - \varepsilon(\tau), \theta_e]$ at the point ϕ_e^0. Let us assume that we have managed to make such a choice of control u, for which the quantity $\varepsilon(t)$ for $t > \tau$ remains positive for all time until the meeting. This will mean that the domain $G^{(1)}[z(\tau), \nu(t), \theta_e]$ all the time until the meeting remains strictly inside the domain $G^{(3)}[y(\tau), \mu(t), \theta_e]$ and, consequently, for a non-increasing moment of absorption $\theta_e(t)$ the meeting will occur not later than at moment $t = \theta_e(\tau)$. It turns out that such a choice of control is possible in the form

$$u[\tau] = u[y(\tau), z(\tau), \mu(\tau), \nu(\tau), \varepsilon(\tau)] = u_s[y, z, \mu, \nu]. \quad (4.1)$$

(The precise meaning of this somewhat conditional notation will be revealed below. (4.1) is considered as a limit for some discrete scheme). This stom algorithm – for calculating the control efforts $u[r]$ can rely only on the values $p(\tau)$, the coitrol efforts $u\ [r]$ can rely only on the values $p(\tau)$, $z(\tau)$, $\mu(\tau)$, $\nu(\tau)$, realized in the process of pursuit. We will call the strategy $u[r]$ (4.1) hereinafter the A-strategy.

Let the touching of the boundaries of regions $G^{(1)}[y - \varepsilon(\tau), \theta_e]$ and $G^{(3)}[z(\tau), \nu(\tau), \theta_e]$ be carried out at the conjugate point φ_e^0, then it is possible to determine the extremal controls, aiming the motion $g(t)$ is $z(t)$ at the point φ_e^0

$$u_0[\tau] = \frac{\mu - \varepsilon}{\xi}\,\omega^0[\tau]. \quad (4.2)$$

The last sentence of the touched uptt of the regularization problem of pursuit.

Figure 8: Figure 8

$$v_0[\tau] = \frac{\dot{\nu}}{\xi} w^0[\tau]. \tag{4.3}$$

Here $\xi(\tau) = \mu(\tau) - \epsilon(\tau) - \nu(\tau)$, and $w^0[\tau]$ is still determined by formyla (1.20) and represents the solution to the problem (1.22), (1.23), (1.24) on limiting speed of response, only now already in (1.23) $\xi(\tau) = \mu(\tau) - \nu(\tau) - \epsilon(\tau)$ and $\theta^0 = \theta_\epsilon$ in (1.24). The time of absorption θ_ϵ is again determined by the smallest positive root of equation (1.17), which we rewrite in the form

$$\kappa(x, \tau, \theta) = \xi(\tau) = \mu(\tau) - \nu(\tau) - \epsilon(\tau), \tag{4.4}$$

where we denote

$$\kappa(x, \tau, \theta) = (D^{-1}c_{[m]}, c_{[m]})^{1/2} = \Big[\int_0^\theta \|w^0[t]\|^2 dt\Big]^{1/2}. \tag{4.5}$$

For further discussion we will need the expression for the derivative

$$\dot{\epsilon}(t) = \dot{\mu}(t) - \dot{\nu}(t) - \dot{\xi}(t), \tag{4.6}$$

calculated along the motion of the system (1.22) for a fixed value $\theta = \theta_c(\tau)$. Differentiating (4.5), we find

$$2\dot{\xi}\dot{\xi} = -\left(D^{-1}\frac{dD}{dt}D^{-1}c_{[m]}, c_{[m]}\right) + 2\left(D^{-1}\frac{dc_{[m]}}{dt}, c_{[m]}\right), \tag{4.7}$$

where in accordance with (1.11) and (1.18)

$$\frac{dD}{dt} = -H^{[m]}H^{[m]\prime}; \quad \frac{dc_{[m]}}{dt} = -H^{[m]}w. \tag{4.8}$$

Let us introduce the notation $\delta w = w - w^0$, then, substituting (4.8) into (4.7) and taking into account (1.20), in final form we obtain

$$\dot{\xi} = -\frac{1}{2\xi}\left[\|w^0\|^2 + 2(w^0, \delta w)\right] \quad (\xi > 0). \tag{4.9}$$

Let further $\delta u = u - u_0$, $\delta v = v - v_0$ — to be the deviations of admissible controls u and v from the extremal strategies u_0 (4.2) and v_0 (4.3) respectively. Taking into account that $w = u - v$, $w^0 = u_0 - v_0$, $\delta w = \delta u - \delta v$, and taking into account (4.6), (3.3) and (4.9), we finally find,

$$\dot{\epsilon} = \frac{\epsilon\|u\|^2}{2\mu(\nu + \xi)} - \frac{\|\delta u\|^2}{2(\nu + \xi)} + \frac{\|\delta v\|^2}{2\nu}. \tag{4.10}$$

Let us now define the desired R-strategy in the form

$$u_c = R[\epsilon, \xi]\, u_0[t], \tag{4.11}$$

where $u_0[t]$ — is the extremal strategy (4.2). We will show that the *regularizing function* $R[\epsilon, \xi]$ in (4.11) can be chosen such that the quantity $\epsilon(t)$ for $t > \tau$ in the pursuit process will remain positive for any δv. Namely, we set

$$R[\epsilon, \xi] = \begin{cases} 1 & \text{for } \xi \geqslant \epsilon^2, \\ \sqrt{\xi}/\epsilon & \text{for } 0 \leqslant \xi < \epsilon^2. \end{cases} \tag{4.12}$$

If $\xi \geqslant \epsilon^2$, then $u_\epsilon = u_0$, $\delta u = 0$, $\dot{\epsilon} \geqslant 0$ and, therefore, the quantity $\epsilon(t)$ does not decrease for any δv.

Figure 9: Figure 9

Let new $0 < \zeta < \varepsilon^2$. Consider on the place $\{\varepsilon, \zeta\}$ the familty of curves, defined by the differential equation

$$\frac{d\varepsilon}{d\zeta} = \frac{\sqrt{\zeta - \varepsilon}}{\varepsilon} \quad (\varepsilon = \varepsilon \text{ when } \zeta = 0). \tag{4.13}$$

The integral curves of this equanition $\varepsilon = f(\zeta)$ for different $\varepsilon = f(0) > 0$ are shown in Fig. 3. Let us calculate the full derivative with respect to time from the function $V(\varepsilon, \zeta) = \varepsilon - f(\zeta)$ in the neighborhood of an arbirsary curve $\varepsilon = f(\zeta)$ in view of differential equations (4.9), (4.10). We will have

$$\frac{dV}{dt} = \frac{\|\alpha^R\|^2}{2\varepsilon\mu\zeta}\,[\mu(1 + \sqrt{\zeta}) - \varepsilon(1 + \mu)] + \frac{1}{2(\nu + \zeta)}\|\delta\mathbf{v} - \delta\mathbf{v}\|^2 + \frac{\zeta}{2\nu(\nu + \zeta)}\|\delta\mathbf{v}\|^2.$$

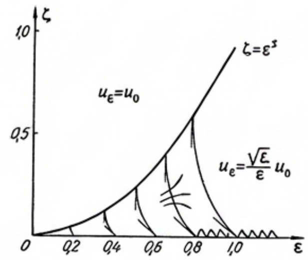

Fig. 3

The quantity $[\mu(1 + \sqrt{\zeta\varepsilon}) - \varepsilon(1 + \mu)]$ is positive at least for sufficiently small μ compared to ε. Therefore, for $0 < \zeta < \varepsilon^2$ the derivative dV/dt-negative, for $0 < \zeta < \varepsilon^2$ the derivative dV/dt is non-negative for any $\delta\mathbf{v}$ and, thus, thus, thus, the trajectories of the ovstem of differential equations (4.9), (4.10) cannot ex intersect the specified curves $\varepsilon = f(\zeta)$ for small values of ε towards the decrease of the value of ε, i.e. $\varepsilon(t)$ for $t > t_0$ will remain non-negative, if only at the initial moment of pursuit $t = t_0$ was $\varepsilon(t_0) > 0$.

It remains to consider the case $\zeta = 0$. By construction of the R-strategy (4.11), (4.12) the function u_ε is continuous for $\zeta > 0$ and for $\zeta = 0$, $u_\varepsilon = 0$. In addition, for $\zeta = 0$ we have $c_{[m]} = 0$, then

$$\zeta' = \lim_{\Delta t \to 0} \frac{\zeta - 0}{\Delta t} = \sqrt{(D^{-1}H^{[m]}\mathbf{v},\, H^{[m]}\mathbf{v})} \tag{4.14}$$

And in coorderance with (4.6):

$$\varepsilon' = \|\mathbf{v}\|^2/2\nu - \zeta. \tag{4.15}$$

Calculating new with account of (4.14), (4.15) the full derivative with respect to time from the indicated above function $V(\varepsilon, \zeta)$, we find, that for $\zeta = 0$

$$\frac{dV}{dt} = -\frac{d\varepsilon}{d\zeta}\,\zeta + \varepsilon = \zeta + \varepsilon = \frac{\|\mathbf{v}\|^2}{2\nu} \geq 0$$

and, consequently, for $\zeta = 0$ the value $\varepsilon(t)$ also remains positive.

Thus, if the pursuer is guided by the specified R-strategy (4.11), (4.12), then the meeting will necessarily take place independently of the actions of the evader no later than the moment $\vartheta_\varepsilon(t_0)$, which serves as the smailest positive root of equation (4.4) for $\tau = t_0$. Until now it was assumed, that the moment of capture $\vartheta_\varepsilon(t)$ remains in the process of pursuit unchanged and equal to $\vartheta_\varepsilon(t_0)$. Havever it is necessary to consider it varying, since the evader can behave unreasonably ($\delta\mathbf{v} \neq 0$) and then the possibility appears to ensure the meeting strictly earlier than the moment $\vartheta_\varepsilon(t_0)$. In the conditions

Figure 10: Figure 10

under the conditions of the introduced regularization, the continuous correction of the absorption moment $\vartheta_\varepsilon(t)$ in accordance with equation (4.4) must be strictly understood as a limiting case of the change of values of ϑ_ε at discrete moments of time t_k $(k = 1, 2, \ldots)$, between which on small intervals $[t_k, t_{k+1} = t_k + \Delta t)$ the control $u_\varepsilon[t]$ is determined for a constant $\vartheta_\varepsilon(t) = \vartheta_\varepsilon(t_k)$, which is the smallest positive root of the equation

$$\varkappa\,[x\,(t_k),\ t_k,\ \vartheta] - \mu\,(t_k) + \nu\,(t_k) + \varepsilon^*\,(t_k) = 0, \tag{4.16}$$

where

$$\varepsilon^*\,(t_k) = \min\,\{\varepsilon\,(t_0);\ \varepsilon\,(t_k) = \mu\,(t_k) - \nu\,(t_k) - \varkappa\,[x\,(t_k),\ t_k,\ \vartheta_\varepsilon\,(t_{k-1})]\}.$$

Using the continuity property of the function $x = \varkappa\,[x\,(\tau),\ \tau,\ \vartheta]$ (4.5) with respect to the variable $\vartheta > \tau$ for any fixed τ, x, $x\,(\tau)$ and taking into account the limit relation $\lim \varkappa\,[x\,(\tau),\ \tau,\ \vartheta] = +\infty$ as $\vartheta \to \tau$, it is possible to establish that always $\vartheta_\varepsilon(t_{k-1}) \geq \vartheta_\varepsilon(t_k)$ $(k = 1, 2, \ldots)$, i.e., the absorption moment, corrected at moments $\tau = t_k$ according to equation (4.16), changes with time, not increasing. In all this, it is important to note that the quantity $\varepsilon\,(t)$ needs to be specified only at the initial moment of pursuit $t = t_0$. Subsequently, it is determined from the relation $\varepsilon\,(t) = \mu\,(t) - \nu\,(t) - \varkappa\,[x\,(t),\ t,\ \vartheta_\varepsilon]$ on the basis of measuring the values $y\,(t)$, $z\,(t)$, $\mu\,(t)$, $\nu\,(t)$. Now, in accordance with (4.16), in formula (4.12) at times $t = t_k$, it is necessary to set $\varepsilon = \varepsilon^*\,(t_k)$.

Let us also note that for $\zeta = 0$, a peculiar sliding mode cue may appear. Indeed, for $\zeta = 0$, the derivative $\dot\zeta$ (4.14) is strictly positive, unless $\nu = 0$. Therefore, in a small time interval $\Delta\, t$, the representative point (ε, ζ) moves from the axis $\zeta = 0$ into the region $0 < \zeta < \varepsilon^2$, where in accordance with (4.9)

$$\dot\zeta = \left(\frac{w^0}{\zeta},\ \nu\right) + o_1\,(\zeta), \quad o_1\,(\zeta) \to 0 \text{ as } \zeta \to 0 \tag{4.17}$$

and, consequently, for $\zeta > 0$, the evader can choose the control ν (for example, $\nu = -\nu_0$) such that ζ will be negative until ζ again turns to zero. Furthermore, taking into account (4.10), (4.17), for $\zeta > 0$ we have

$$\frac{d\dot\zeta}{d\varepsilon} = \frac{\left(\frac{w^0}{\zeta},\ \nu\right)}{-\left(\frac{w^0}{\zeta},\ \nu\right) + \frac{\|\nu\|^2}{2\nu}} + o_2\,(\zeta), \tag{4.18}$$

$$o_2\,(\zeta) \to 0 \text{ as } \zeta \to 0$$

and, taking into account (4.14), (4.15), for $\zeta = 0$ we get

$$\frac{d\zeta}{de} = \frac{\dot\zeta}{-\dot\zeta + \frac{\|\nu\|^2}{2\nu}}. \tag{4.19}$$

From (4.18) and (4.19) it follows that for eany ν, for which $\left(\frac{w^0}{\zeta},\ \nu\right) \leqslant 0$, and as $\zeta \to 0$, the inequality $0 > \frac{d\zeta}{de} > -1$, and for $\zeta = 0$, the tangent to the courve $\zeta = \zeta\,(\varepsilon)$ cannot form with the oxis $\zeta = 0$ an angle greater than

Figure 11: Figure 11

than 135°. This means that the possible sliding mode, which in Fig. 3 is schematically represented by the sawtooth curve, leads only to an increase in $\varepsilon(t)$.

Thus, from the output one can conclude that the control u_ε (4.11), (4.12) provides the maximum that the pursuer can achieve due to an arbitrarily small increase in the control resource μ, regularizing the problem of pursuit under the condition of coincidence of the selected coordinates at the moment of meeting θ_ε, which in regular cayes is slightly different from the moment of absorption θ_0.

However, as numerical experiments show, the constructed R-strategy (4.11), (4.12), generally speaking, is little suitable for practical implemenations of pursuit processes on computational devices. From this point of view, the choice of the regularizzing function $R[\varepsilon, \zeta]$ is more ydowfnent in the form

$$R[\varepsilon, \zeta] = \begin{cases} 1 & \text{for } \zeta \geqslant \varepsilon, \\ \zeta/\varepsilon & \text{for } 0 \leqslant \zeta < \varepsilon. \end{cases} \tag{4.20}$$

Let us establish that in the case of the pursuer using the R-strategy (4.11), (4.20) for sany δv, the function $\varepsilon(t)$, although it can deneuse, cannot turn to nyro in a finite time and therefore will remain a positive quantitity during the entire pursuit time. For this purpose, let us consider the homogoneous differential equation

$$\frac{d\varepsilon}{d\zeta} = \frac{\zeta - \varepsilon}{\varepsilon} \quad (\varepsilon = \varepsilon \text{ for } \zeta = 0). \tag{4.21}$$

The family of integral curves of this equation has the form

$$(\varepsilon^2 + \varepsilon\xi - \zeta^2)^{1/2} \left(\frac{\varepsilon + a\zeta}{\varepsilon + b\zeta}\right)^{\frac{1}{2(a-b)}} = \varepsilon, \tag{4.22}$$

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}.$$

Let us compute now in the region $0 < \zeta < \varepsilon$ the total time derivative of the function $V(\varepsilon, \zeta)$, for which the integral curves $\varepsilon = f(\zeta)$ of equation (4.21), coornerstrying to different values of ε, are rcurves of constoant lpnstnosl $V(\varepsilon, \zeta) = \varepsilon$. In coorderance with (4.22)

$$V(\varepsilon, \zeta) = (\varepsilon^2 + \varepsilon\xi - \zeta^2)^{1/2} \left(\frac{\varepsilon + a\zeta}{\varepsilon + b\zeta}\right)^{\frac{1}{2(a-b)}}. \tag{4.23}$$

Consitering (4.9), (4.10) and (4.20), we obtain

$$\frac{dV}{dt} = \frac{\partial V}{\varepsilon^2 + \varepsilon\xi - \zeta^2}\left(\varepsilon - \frac{\zeta - \varepsilon}{\varepsilon}\zeta\right) =$$

$$= \frac{\partial V}{\varepsilon^2 + \varepsilon\xi - \zeta^2}\left[\frac{\|w^0\|^2}{2\zeta^2}\left[\frac{\zeta^2(\mu - \varepsilon)}{\varepsilon\mu} + \frac{(\zeta - \varepsilon)^2(\mu - \varepsilon)}{\varepsilon^2} = \right.\right.$$

$$\left.\left. - \frac{(\varepsilon - \zeta)\zeta}{\varepsilon}\right] + \frac{\|\delta v\|^2}{2v} - \frac{(\zeta - \varepsilon)}{\varepsilon\zeta}(w^0, \delta v)\right]. \tag{4.24}$$

Figure 12: Figure 12

Let us estimate in the region $0 < \ll \varepsilon$ the derivative dV (4.24) from below. Taking into account that $\|w^0\|^2/\zeta^2 = k$ is a limited value, and considering that

$$\min_{\delta v} \left[\frac{\|\delta v\|^2}{2v} - \frac{(\zeta - \varepsilon)}{\varepsilon\zeta} (w^0, \delta v) \right] = - \frac{v(\zeta-\varepsilon)^2}{2\varepsilon^2\zeta^2} \|w^0\|^2,$$

$$\frac{5}{4}\varepsilon^2 > (\varepsilon^3 + \varepsilon\zeta - \zeta^2) > \varepsilon^2 \quad \text{for } 0 < \zeta \ll \varepsilon,$$

we finded for $\zeta \ll \varepsilon \ll \mu$

$$\frac{dV}{dt} \geqslant -\frac{k}{2} V,$$

but in the region $0 < \zeta \ll \varepsilon$ we have

$$\varepsilon \left(\frac{3+\sqrt{5}}{3-\sqrt{5}} \right)^{\frac{1}{2\sqrt{5}}} \geqslant V(\varepsilon, \zeta) > \varepsilon,$$

consequently,

$$\varepsilon(t) \left(\frac{3+\sqrt{5}}{3-\sqrt{5}} \right)^{\frac{1}{2\sqrt{5}}} \geqslant V(t) \geqslant V(t_0)\, e^{-\frac{k}{2}t}$$

As a result we get $\varepsilon(t) \geqslant \alpha e^{-\frac{k}{2}t}$ $(\alpha > 0)$, I.e. the fynction $\varepsilon(t)$ for $0 < \zeta \ll \varepsilon$ remains positive nutive for all time before the meeting. Ecnu if $\zeta > \varepsilon$, then, as in the case (4.11), (4.12), we have if $\zeta > \varepsilon$ and, consequently, in the region $\zeta > \varepsilon$ function $\varepsilon(t)$ does not dencrease. Finally, using (4.15) and (4.24), it is clearly obtained in unoreased, that $e(t)$ cannot dencrease also in the case $\zeta = 0$.

Remark 1. Thus, we see, that at least for theoretical purposes, the choice of the regular vecred andop peryluriusupyronell fymction $R[s, \zeta]$ is not uncue ediqueretable. From a practical point of view, the most ydodhient would be the fynction

$$R[\varepsilon, \zeta] = \begin{cases} 1 & \text{for } \zeta > 0, \\ 0 & \text{for } \zeta = 0. \end{cases} \tag{4.25}$$

However, unfortunately, in this clase the possibility arrives of the ensurkence of a cloikizing mode, analorious to the one mentioned abrove for the R-strategy (4.11), (4.12), hot reading to him to a reuped denerease in the value of $\varepsilon(t)$. In fact, fart, for $u_s = R[e, \zeta]\, u_0$, where $R[e, \zeta]$ is determined by the formyle (4.25), we have in the case $\zeta > 0$

$$\frac{d\zeta}{de} = \frac{\left(\frac{w^0}{\zeta}, \delta v \right)}{\frac{\varepsilon(v+\zeta)}{2\mu\zeta^2} \|w^0\|^2 + \frac{\|\delta v\|^2}{2v}} + o_S(\zeta), \tag{4.26}$$

$$o_S(\zeta) \to 0 \text{ for } \zeta \to 0$$

etc., for example, for $v = 0$ $\lim\limits_{\zeta \to 0} \frac{d\zeta}{de} = \frac{2\mu}{\varepsilon + \mu} < -1 - \gamma_1$, as long as $0 < \varepsilon < \mu - \gamma_2$ $(\gamma_1 > 0,\ \gamma_2 > 0)$. At the same time for $\zeta = 0$ the value $d\zeta/de$ (4.19) for soctatovnly small values $\|v\|$ turns out to be aproxessally close to -1. Thus, ecrn $v = \gamma = \text{const}$, where $\|v\|$ is suctatiently small, then a specific clorizing mode canot aurise, able to irward even before the sceeting to the elimination of the e-interlaye between ebictions $G^{(1)}[y(\tau), \mu(\tau), v_c]$ and $G^{(2)}[z(\tau),$

Figure 13: Figure 13