ON THE SOLVABILITY OF THE SYSTEM OF EQUATIONS OF STRONG BENDING OF PLATES

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Abstract

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Yu. A. DUBINSKII

ON THE SOLVABILITY OF THE SYSTEM OF EQUATIONS OF STRONG BENDING OF PLATES

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In this paper the existence is proved of both a generalized and a smooth solution of system (1), which describes the strong bending of a thin plate; moreover, for proving the existence of a generalized solution only the simplest energy estimate and weak convergence in L_2 are used. We note that the existence of a generalized solution for A. Föppl' s equations, which are a consequence of (1), was obtained earlier by other methods and under stronger assumptions, for example, in papers ([2],[3]).

As is known (see, for example, ([1])), the complete system of equations of strong bending of a plate has the form

$$\frac{h^2E}{12(1-\sigma^2)}\Delta^2\zeta - h\frac{\partial}{\partial x_\beta}\left(\sigma_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\right) = P(x,y), \qquad \frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} = 0. \tag{1}$$

Here h is the thickness of the plate; E is Young' s modulus; σ $(0 < \sigma < 1)$ is Poisson' s ratio; $\sigma_{\alpha\beta}$ is the stress tensor, related to the strain tensor $u_{\alpha\beta}$ by the formulas

$$\sigma_{\alpha\beta} = E(1-\sigma^2)^{-1}[(1-\sigma)u_{\alpha\beta} + \sigma\delta_{\alpha\beta}u_{\gamma\gamma}] \tag{2}$$

$(\alpha, \beta$, and γ run through the two values x and $y)$.

Further,

$$u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) + \frac{1}{2}\frac{\partial\zeta}{\partial x_\alpha}\frac{\partial\zeta}{\partial x_\beta}, \tag{3}$$

where $\mathbf{u} = (u_x, u_y)$ is the strain vector; $\zeta(x,y)$ is the vertical displacement of the point (x,y) of the plate. Thus the unknowns in (1) are the three functions $u_x(x,y)$, $u_y(x,y)$, and $\zeta(x,y)$, defined in the domain G occupied by the plate.

Let the boundary conditions for system (1) have the form

$$\zeta(x,y) = u_x(x,y) = u_y(x,y) = 0, \qquad \frac{\partial \zeta}{\partial x}(x,y) = \frac{\partial \zeta}{\partial y}(x,y) = 0, \qquad \forall (x,y) \in \partial G, \tag{4}$$

which corresponds to the case of a clamped plate.

Definition. A **generalized solution** of problem (1), (4) is a system of functions $(\zeta, \mathbf{u})$ with finite energy integral

$$E(\zeta, \mathbf{u}) = \iint_G [(\Delta\zeta)^2 + \sigma_{\alpha\beta} u_{\alpha\beta}]\, dx\, dy^*,$$

satisfying the integral identity

$$\frac{h^3 E}{12(1-\sigma^2)} \langle \Delta\xi, \Delta\tilde{\xi} \rangle + h \left\langle \sigma_{\alpha\beta} \frac{\partial \xi}{\partial x_\alpha}, \frac{\partial \tilde{\xi}}{\partial x_\beta} \right\rangle = \langle P, \tilde{\xi} \rangle,$$

$$\left\langle \sigma_{\alpha\beta}, \frac{\partial v_\alpha}{\partial x_\beta} \right\rangle = 0, \qquad \forall \tilde{\xi}(x,y),\ v_x(x,y),\ v_y(x,y) \in C_0^\infty(G) \tag{3'}$$

$$\left(\langle u, v \rangle = \iint_G u \cdot v \, dx\, dy; \right.$$

in the last two equations summation is only over β) and satisfying the boundary values (4) in the mean (^{5}).

Theorem 1. If $P(x,y) \in W_2^{(-2)}$, then problem (1), (4) has at least one generalized solution.

Proof. Let $(\xi^k(x,y),\ v_x^k(x,y),\ v_y^k(x,y))$ be a system of smooth functions finite in the domain G, complete, for example, in L_2. The approximate solution (ξ^n, u_x^n, u_y^n) is sought in the form

$$(\xi^n, u_x^n, u_y^n) = \sum_{k=1}^{n} (c_{kn}^1 \xi^k,\ c_{kn}^2 v_x^k,\ c_{kn}^3 v_y^k);$$

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[*] The integral $E(\zeta, \mathbf{u})$ is equivalent to the mechanical energy of the plate E_m in the sense that $E_m \le c_1 E \le c_2 E_m$, where $c_1, c_2 > 0$ are constants. In addition, we note that from the purely mathematical point of view system (1) is of interest because, besides the principal part, lower-order terms play an essential role in determining the class of generalized solutions.

the constants c^i_{kn}, $i = 1, 2, 3$, are determined from the system of moment equations

$$\frac{h^3 E}{12(1-\sigma^2)}\langle\Delta\xi^n, \Delta\xi^k\rangle + h\left\langle\sigma^n_{\alpha\beta}\frac{\partial\xi^n}{\partial x_\alpha}, \frac{\partial\xi^k}{\partial x_\beta}\right\rangle = \langle P, \xi^k\rangle, \tag{5}$$

$$\left\langle\sigma^n_{\alpha\beta}, \frac{\partial v^k_\alpha}{\partial x_\beta}\right\rangle = 0, \qquad k = 1, \dots, n,$$

where $\sigma^n_{\alpha\beta}$ are defined by formulas (2), (3) with $(\xi, \mathbf{u})$ replaced by $(\xi^n, \mathbf{u}^n)$. The solvability of system (5) follows from a lemma of M. I. Vishik (^{4}); moreover, for the approximate solutions the a priori estimate

$$E(\xi^n, \mathbf{u}^n) \leq K_1 \|P\|_{W_2^{(-2)}}, \tag{6}$$

holds, where K_1 does not depend on $(\xi^n, \mathbf{u}^n)$. (To obtain estimate (6), it is enough to multiply (5) by c_{in}, sum over k, then add the resulting equalities and take into account the symmetry of the tensor $u_{\alpha\beta}$; the right-hand side of (5) is estimated with the aid of Schwarz' s inequality.) From estimate (6) and the known inequalities for the Laplace operator we obtain that $\xi^n(x, y) \in W_2^{(2)}$, with $\|\xi^n\|_{W_2^{(2)}} \leq K_2$. In addition, it follows from (6) that $\langle\sigma^n_{\alpha\beta}, u^n_{\alpha\beta}\rangle \leq K_3$, or, equivalently,

$$\frac{E}{1-\sigma^2}\iint_G [(u^n_{xx})^2 + 2\sigma u^n_{xx} u^n_{yy} + (u^n_{yy})^2]\,dx\,dy + \frac{2E}{1+\sigma}\iint_G (u^n_{xy})^2\,dx\,dy \leq K_3,$$

after which it follows from formulas (3) that $\partial u^n_x/\partial x \in L_2$, $\partial u^n_y/\partial y \in L_2$, $\partial u^n_x/\partial y + \partial u^n_y/\partial x \in L_2$, and they range over a bounded set in L_2. Hence, it may be assumed (possibly after selecting a subsequence) that $\xi^n \to \xi$ weakly in $W_2^{(2)}$; $\mathbf{u}^n \to \mathbf{u}$, $\partial u^n_x/\partial x \to \partial u_x/\partial x$, $\partial u^n_y/\partial y \to \partial u_y/\partial y$, and $\partial u^n_x/\partial y + \partial u^n_y/\partial x \to \partial u_x/\partial y + \partial u_y/\partial x$ weakly in L_2, where $(\xi, \mathbf{u})$ is some system of functions with finite energy E and satisfying the boundary conditions (4) in the mean.

We shall show that $(\xi, \mathbf{u})$ is the desired generalized solution. To this end, note that, by virtue of the completeness in L_2 of the system (ξ^k, v^k_x, v^k_y), it is enough to prove that $(\xi, \mathbf{u})$ satisfies relations ($3'$), where $(\tilde{\xi}, v_x, v_y)$ are replaced by (ξ^k, v^k_x, v^k_y) for any $k = 1, 2, \dots$.

In other words, it is enough to prove the possibility of passing to the limit in (5) as $n \to \infty$ with fixed k. In the linear term $\langle\Delta\xi^n, \Delta\xi^k\rangle$ this is obvious. Further, from the weak convergence $\xi^n \to \xi$ in $W_2^{(2)}$ it follows that

$$\xi^n \to \xi \quad \text{strongly in } W_p^{(1)} \ (p > 1 \text{ arbitrary}), \qquad \frac{\partial\xi^n}{\partial x_\alpha}\frac{\partial\xi^n}{\partial x_\beta} \to$$

$$\to \frac{\partial\zeta}{\partial x_\alpha} - \frac{\partial\zeta}{\partial x_\beta}$$

strongly in L_2. Consequently, $u^n_{\alpha\beta} \to u_{\alpha\beta}$ weakly in L_2, and since $\sigma^n_{\alpha\beta}$ is expressed linearly in terms of $u^n_{\alpha\beta}$, it follows that $\sigma^n_{\alpha\beta} \to \sigma_{\alpha\beta}$ weakly in L_2. This justifies the passage to the limit in the last two equations of system (3).

Lemma. If $u_n \to u$ strongly in L_2, and $v_n \to v$ weakly in L_2, then $u_n v_n \to uv$ weakly in L_1.

For the proof see (6).

From this lemma we obtain that

$$\left\langle \sigma^n_{\alpha\beta} \frac{\partial \zeta^n}{\partial x_\alpha}, \frac{\partial \zeta^k}{\partial x_\beta} \right\rangle \to \left\langle \sigma_{\alpha\beta} \frac{\partial \zeta}{\partial x_\alpha}, \frac{\partial \zeta^k}{\partial x_\beta} \right\rangle, \qquad k = 1, 2, \ldots$$

As a result we obtain that $(\zeta, \mathbf{u})$ is a solution of problem (1), (4). Theorem 1 is proved.

Remark. The membership $P(x,y) \in W_2^{(-2)}$ means that $P(x,y)$ has the form

$$P(x,y) = \sum_{|\alpha| \le 2} D^\alpha f_\alpha(x,y),$$

where $f_\alpha(x,y) \in L_2$

$$\left(D^\alpha = \frac{\partial^\alpha}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2 \right),$$

i.e. $P(x,y)$ may be, in particular, a generalized function of the type $\delta(x,y)$ and its derivatives. This corresponds to loads on the plate concentrated at individual points. For such loads it is impossible to obtain a smoother solution than in Theorem 1.

Theorem 2 (smoothness theorem). Let $P(x,y) \in L_p$, where $p < 2$ is arbitrary. Then the generalized solution $(\zeta, \mathbf{u})$ is such that

$$\zeta \in W_p^{(4)} \cap \overset{\circ}{W}{}_2^{(1)}, \qquad \mathbf{u} \in W_p^{(2)} \cap \overset{\circ}{W}{}_2^{(1)}.$$

Proof. Consider the last two equations in (1):

$$\frac{E}{1-\sigma^2} \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \sigma \frac{\partial u_y}{\partial y} \right) + \frac{E}{2(1+\sigma)} \frac{\partial}{\partial y} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + f_1(\zeta) = 0, \tag{7}$$

$$\frac{E}{2(1+\sigma)} \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \frac{E}{1-\sigma^2} \frac{\partial}{\partial y} \left(\frac{\partial u_y}{\partial y} + \sigma \frac{\partial u_x}{\partial x} \right) + f_2(\zeta) = 0,$$

where

$$f_1(\zeta) = \frac{E}{2(1-\sigma^2)}\frac{\partial}{\partial x}\left[\left(\frac{\partial\zeta}{\partial x}\right)^2 + \sigma\left(\frac{\partial\zeta}{\partial y}\right)^2\right] + \frac{E}{2(1+\sigma)}\frac{\partial}{\partial y}\left(\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\right),$$

and $f_2(\zeta)$ coincides with $f_1(\zeta)$ after interchanging x and y.

It is easily verified that system (7) is elliptic (indeed, strongly elliptic). Further, since $\zeta \in \overset{\circ}{W}{}_2^2$, it follows that $f_1(\zeta), f_2(\zeta) \in L_p$, where $p < 2$ is arbitrary. Consequently, by the smoothness theorem for linear elliptic systems (see, for example, (7)), $\mathbf{u}(x,y) \in W_p^{(2)} \cap \overset{\circ}{W}{}_2^{(1)}$.

Further, since $\partial\sigma_{\alpha\beta}/\partial x_\beta = 0$, the first equation in (1) can be written in the form

$$-\frac{h^3E}{12(1-\sigma^2)}\Delta^2\zeta - h\sigma_{\alpha\beta}\frac{\partial^2\zeta}{\partial x_\alpha \partial x_\beta} = P(x,y), \tag{8}$$

where, since $\zeta \in W_2^{(2)}$ and $\mathbf{u} \in W_p^{(2)}$, $p < 2$ arbitrary, we have

$$\sigma_{\alpha\beta}\frac{\partial^2\zeta}{\partial x_\alpha \partial x_\beta} \in L_p, \quad p < 2$$

arbitrary. Taking into account that, by assumption, $P(x,y) \in L_p$, from equation (8) we obtain that the solution $\zeta(x,y) \in W_p^{(4)} \cap W_2^{(2)}$. Theorem 2 is proved.

Further analogous arguments lead to the following result.

Theorem 3. If $P(x,y) \in W_p^{(s)}$, $p > 1$, then the solution $(\zeta, \mathbf{u})$ is such that $\zeta(x,y) \in W_p^{(s+4)} \cap \dot{W}_2^{(1)}$, $\mathbf{u}(x,y) \in W_p^{(s+2)} \cap \dot{W}_2^{(1)}$.

Corollary. If $P(x,y)$ is an infinitely differentiable function, then the solution $(\zeta, \mathbf{u})$ is also infinitely differentiable.

Moscow Power Engineering
Institute

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Note: Figure translations are in progress. See original paper for figures.