ON MIXED DERIVATIVES OF FUNCTIONS SUMMABLE WITH A WEIGHT

If

1967

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. V. USPENSKII

ON MIXED DERIVATIVES OF FUNCTIONS SUMMABLE WITH A WEIGHT

(Presented by Academician S. L. Sobolev on 27 IV 1966)

In the present paper we study classes of functions that are defined in a smooth domain G and have in it nonmixed derivatives, summable with a weight, generally speaking of different orders with respect to different variables. The principal problem solved here is the description of those functional spaces to which the mixed derivatives on G belong. For functions belonging on R^n to the space $W_p^{l_1 \dots l_n}(G)$ of Sobolev, this question was solved by L. N. Slobodetskii ([1]), who formulated without proof the following theorem:

If

$$f \in W_p^{l_1 \dots l_n}(R^n), \qquad 1 < p < \infty, \qquad \sum_{i=1}^{n} \frac{k_i}{l_i} \leqslant 1, \tag{1}$$

then

$$D^k f \in L_p(R^n), \qquad \text{where } |k| = \sum_{i=1}^{n} k_i.$$

From the examples constructed (see, for instance, ([2])) it follows that the restriction $1 < p < \infty$ is essential, since for $p = 1$ and $p = \infty$ theorem (1) ceases to be true.

The case when G is a domain with a curvilinear boundary proved to be more complicated. Here one should note a result of A. P. Calderón ([3]), who extended theorem (1), for $l = l_i$, $i = 1, 2, \dots, n$, to a broad class of domains. However, from examples constructed by S. M. Nikol' skii and V. P. Il' in it follows that for $l_i \neq l_j$ this result can no longer be extended even to the case of analytic domains.

Thus, if one restricts oneself to considering the cases in which the embedding theorems for R^n can be transferred to G, then one can pose only the question of finding, for each class of functions, its own family of domains. The most complete result of this kind was recently obtained for the spaces $W_p^{l_1 \dots l_n}(G)$ by O. V. Besov and V. P. Il' in. In the present paper theorem (1) is generalized to the case of functions summable with a certain weight that degenerates on the boundary of the domain, and a class of domains with smooth boundary is singled out for which this theorem holds. The paper also contains a number of results developing the author' s work ([4]).

Below we shall assume that G has the form

$$\varphi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \leqslant x_j \leqslant A + \varphi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$
$$a_i \leqslant x_i \leqslant b_i, \qquad i = 1, 2, \dots, n;\ i \neq j, \tag{2}$$

where $\varphi \in C^l,\ |D^k\varphi| < M,\ |k| > 0,\ A > 0$.

Let

$$\rho = \rho(x, \partial G) \leqslant c|x_j - \varphi| \leqslant c_1\rho(x, \partial G).$$

We define the space $W^{l_1 \dots l_n}_{p,\alpha_1 \dots \alpha_n}(G) = W^l_{p,\alpha}(G)$ as the closure of C^l

by the norm

$$\|f\| W^l_{\alpha,p}(G) = \|f\|_{L_p(G)} + \sum_{i=1}^{n} \left\| \rho^{\alpha_i/p} \frac{\partial^{l_i} f}{\partial x_i^{l_i}} \right\|_{L_p(G)}.$$

In what follows we shall assume $j = n$. Denote $\lambda_i = l_i - \alpha_i/p$, $i = 1, 2, \dots, n$.

Theorem 1. *Let $f \in W^l_{p,\alpha}(G)$, $p > 1$, and $\lambda_n \geq \lambda_i$, $a_n \geq \alpha_i$, $i = 1, 2, \dots, n$.*

Then the transformation $f(x_1, \dots, x_n) \to f(x'_1, \dots, x'_{n-1}, \varphi(x'_1, \dots, x'_{n-1}) + x'_n)$, where $\varphi \in C^l$, $x \in G$, $x' \in G'$, is a linear operator $W^l_{p,\alpha}(G) \to W^l_{p,\alpha}(G')$, where G' is the image of G.

We note that Theorem 1 also holds for the classes of functions $\beta^l_p(G)$, $H^l_p(G)$.

Theorem 2. *Let $f \in W^l_{p,\alpha}(G)$, $1 < p < \infty$. Put*

$$\gamma_i = (\lambda_n - \lambda_i)/l_i + 1, \quad i = 1, 2, \dots, n; \qquad \beta = \sum_{i=1}^{n} k_i\gamma_i - \lambda_n,$$

$$\varkappa_i = l\gamma_i - \sum_{i=1}^{n} k_i\gamma_i,$$

where k_i are integers.

Then, if

$$\lambda_n - \lambda_i \geq 0, \qquad \varkappa_i \geq 0, \quad i = 1, 2, \dots, n; \qquad \beta > -1/p,$$

then

$$\rho^\beta D^k f \in L_p(G).$$

We outline a brief proof. Let

$$K_0 = \prod_{i=1}^{n} \frac{u^{\gamma_i}}{x_i^2 + u^{2\gamma_i}}.$$

Then one can choose constants c_ν, $\nu = 0, 1, \dots, N$, such that the function

$$K(x,u) = \sum_{\nu=0}^{N} c_\nu u^\nu \frac{\partial^\nu K_0}{\partial u^\nu}$$

has the following properties:

$$\frac{\partial K}{\partial u} = \sum_{i=1}^{n} \frac{\partial^{l_i} \widetilde{K}_i(x,u)}{\partial x_i^{l_i}},$$

$$\left| D^k \widetilde{K}_i(x,u) \right| \le C u^{l_i\gamma_i - 1 - \sum_{i=1}^{n} k_i\gamma_i} K_0(x,u), \qquad |k| = \sum_{i=1}^{n} k_i.$$

Moreover, $c_0 > 0$ (see ([10])).

Continuing the function $f(x_1, \dots, x_{n+1}, x_n + \varphi(x_1, \dots, x_{n-1}))$ across the plane $x_n = 0$ by the method of H. Whitney and M. Hestenes (see ([5])), one can represent the function f in the form

$$f = \int_0^\infty \int_{R^n} f(t) \frac{\partial K}{\partial v}(x - t, v)\, dt\, dv.$$

Then, using the properties of the kernel K and generalizing a theorem of Calderón and Zygmund ([6]) on the boundedness of a singular operator in L_p, we obtain the assertion of Theorem 2.

From Lemma 2 ([11]) and Theorem 1 one can obtain the following embedding theorem, which generalizes the corresponding result ([7]).

Theorem 3. Let $f \in W^l_{p,\alpha}(G)$, $\lambda_n \geqslant \lambda_i > 0$, $p > 1$. Denote by R_n^+ the half-space $x_n \geqslant 0$; $W_p^r(R_n^+)$ is the generalized Sobolev space with fractional index.

Then, if $a_n > 0$, $a_i \geqslant 0$, $i = 1, 2, \dots, n-1$, the function

$$F(x) = f(x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_{n-1})) \in W_p^r(R_n^+)$$

and

$$\|F\|_{W_p^r(R_n^+)} \leqslant c\|f\|_{W_p^l(G)}$$

where

$$r_i = \frac{l_i\lambda_n}{\lambda_n + a_i/p}, \qquad i = 1, 2, \dots, n.$$

From Theorem 3, using the results of the work ([8]), one can obtain, in terms of the classes W_p^r and B_p^r, various embedding theorems for the weighted classes $W_{p,\alpha}^l(G)$; however, the author will not dwell on this. Here it is important to note that the condition $\lambda_n \geqslant \lambda_i$ in all cases ensures the preservation, for the classes of functions $W_{p,\alpha}^l(G)$ (and also $B_p^r(G)$, $H_p^r(G)$), of all embedding theorems that hold for the whole space (half-space). Therefore it is natural to call such domains regular. We note that the class of regular domains considered in this work contains, for $a_i \neq 0$, a number of new cases not considered by V. P. Il' in in ([9]).

All the results of the work also carry over to domains that are finite unions of domains of the form (2).

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
24 IV 1966

REFERENCES

1. L. N. Slobodetskii, DAN, **120**, No. 3 (1958).
2. B. S. Mityagin, DAN, **123**, No. 4 (1958).
3. A. P. Calderon, Conference of Partial Differential, University of California, 1960.
4. S. V. Uspenskii, DAN, **164**, No. 4 (1965).
5. G. M. Fikhtengol' ts, *Course of Differential and Integral Calculus*, 1, Moscow–Leningrad, 1958.
6. A. P. Calderon, A. Zygmund, Am. J. Math., **78**, No. 2 (1956).
7. Ya. S. Bugrov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **77** (1965).
8. O. V. Besov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **40** (1961).

9. V. P. Il' in, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **84** (1965).

10. S. V. Uspenskii, Sibirsk. matem. zhurn., **7**, No. 2, 409 (1966).

11. S. V. Uspenskii, Sibirsk. matem. zhurn., **7**, No. 3, 650 (1966).

Note: Figure translations are in progress. See original paper for figures.