Reports of the Academy of Sciences of the USSR

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Volume 174, No. 6

UDC 517.933

MATHEMATICS

F. M. KIRILLOVA, S. V. CHURAKOVA

RELATIVE CONTROLLABILITY OF LINEAR DYNAMICAL SYSTEMS WITH DELAY

(Presented by Academician L. S. Pontryagin on 25 VIII 1966)

Stationary systems.

1. The determining equation of the control system.

Let the motion of a point $x(t)$ in an n-dimensional space X be given by the vector equation

$$\dot{x}(t) = Ax(t) + Bx(t-h) + cu(t), \qquad t \geq t_0, \tag{1}$$

$$z(t_0) = \{x(t_0),\, x(s),\, t_0 - h \leq s < t_0\} =$$

$$= \{x_0,\, \varphi(s),\, t_0 - h \leq s < t_0\} = z_0,$$

where A, B are constant matrices; c is a constant vector; h is a positive number (the delay); $u(t)$ is the control; $z_0 = \{x_0, \varphi(s)\}$ is the initial state; $\varphi(s)$ is a piecewise-continuous function.

Denote by U_T the class of piecewise-continuous functions $u(t)$ given on the interval $I_T = [t_0, t_0 + T]$. The totality U_T^1 of functions $u(t) \in U_T$ constrained by the condition

$$\max_{t \in I_T} |u(t)| \leq 1, \tag{2}$$

will be called the set of admissible controls of (1).

To the system (1) we assign the equation

$$q_k(l) = Aq_{k-1}(l) + Bq_{k-1}(l-1), \tag{3}$$

$$q_1(1) = c, \qquad q_k(0) = 0, \qquad q_0(l) = 0, \qquad l = 1, 2, \dots; \quad k = 1, 2, \dots,$$

which we shall call the determining equation of the control system (1). The sequence Q_α^β, $Q_\alpha^\beta = \{q_k(l),\ l = 1, 2, \dots, \alpha;\ k = 1, 2, \dots, \beta\}$, is the solution of equation (3) defined on the set $1 \le l \le \alpha,\ 1 \le k \le \beta$.

Lemma 1. The rank of $\{Q_\alpha^\beta,\ \alpha = 1, 2, \dots;\ \beta = 1, 2, \dots\}$ is equal to the rank of $\{Q_\alpha^n,\ \alpha = 1, 2, \dots\}$.

Denote Q_α^n by Q_α. The determining equation will be called nondegenerate for a given α if the sequence Q_α has rank n. The determining equation will be called nondegenerate if it is nondegenerate for at least one $\alpha,\ \alpha \ge 1$. We shall say that the control system is normal (2) if the determining equation is nondegenerate for $\alpha = 1$.

We represent the solution $x_T = x(z_0, u, t)\big|_{t=t_0+T}$ of equation (1) in the form

$$x_T = c_T + S_{Tu},$$

where

$$c_T = P_{z_0} = F(t_0 + T, t_0)x_0 + \int_{t_0-h}^{t_0} F(t_0 + T, \tau + h)B\varphi(\tau)\,d\tau,$$

$$S_{Tu} = \int_{t_0}^{t_0+T} F(t_0 + T, \tau)cu(\tau)\,d\tau,$$

$$\partial F(t, \tau)/\partial\tau = -F(t, \tau)A - F(t, \tau + h)B, \qquad F(t, t) = E,$$

$$F(t, \tau) \equiv 0, \qquad \tau > t.$$

2. Relative controllability. Definitions.

A state z_0 is **relatively controllable on** I_T if there exists a control $\bar{u} \in U_T$ such that $x_T = x(z_0, \bar{u}, t_0 + T) = 0$. If every initial state z_0 of system (1) is relatively controllable on I_T, then the **system is relatively controllable on** I_T. A **relatively controllable system** is system (1) for which, for every state z_0, there exist $T < +\infty$ and $u \in U_T$ such that $x(z_0, u, t_0 + T) = 0$.

Theorem 1. System (1) is relatively controllable on I_T if and only if the determining equation of the system is nondegenerate for
$m = [T/h] + 1$.

Corollary. In order that system (1) be relatively controllable, it is necessary and sufficient that the determining equation be nondegenerate.

Definitions. A state z_0 **has an admissible control on** I_T if there exists $\tilde{u} \in U_T^1$ such that $x(z_0, \tilde{u}, t_0 + T) = 0$. A state z_0 **has an admissible control** if z_0 satisfies the preceding definition for some T, $T < +\infty$.

Theorem 2. The initial state z_0 of system (1) has an admissible control on I_T if and only if

$$\delta(t_0, T) = \max_{\|g\| \leq 1} [(g, c_T) - \|S_T^* g\|] = (g_T^0, c_T) - \|S_T^* g_T^0\| \leq 0,$$

$$g \in X^*, \qquad \|g\|^2 = \sum g_i^2.$$

Theorem 3. If system (1) is relatively controllable on I_T, then there exists $\varepsilon > 0$ such that every initial state from the set $\|z_0\| \leq \varepsilon$ has an admissible control on I_T.

Remark 1. $\varepsilon = \varepsilon(t_0) = \mu/\lambda$, $\lambda = \max_z \|Pz\|/\|z\|$, $\mu = \min_g \|S_T^* g\|/\|g\|$.

Corollary. If system (1) is relatively controllable, then there exists an ε-neighborhood of the zero state, each point of which has, for at least one τ, $\tau \leq nh$, an admissible control on I_T with $T \geq \tau$.

3. Existence of an optimal control.

We shall call the number T^0 the **optimal time** for z_0 if

$$T^0 = \inf_{u \in U'_T} \{T : x(z^0, u, T) = 0\}.$$

The control u^0 corresponding to T^0 will be called **optimal**.

Lemma 2. The function $\delta(t_0, T)$ is continuous in T (cf. (3)).

Theorem 4. If the state z_0 has an admissible control, then there also exists an optimal control for z_0. The optimal time T^0 is equal to the smallest root of the equation $\delta(t_0, T) = 0$. The optimal control u^0 satisfies the condition

$$(S_{T^0}^* g_{T^0}^0, u^0) = \min_{u \in U'_{T^0}} (S_{T^0}^* g_{T^0}^0, u).$$

Remark 2. If U_T^1 is specified by means of (2), then

$$u^0(t) = -\operatorname{sign} \psi(t), \tag{4}$$

where $\psi(t)$ is a solution of the equation

$$\sum_{i=0}^{n}\sum_{j=0}^{i} p_{ij}\psi^{(n-i)}(\tau+jh)=0, \qquad \tau\in I_T, \qquad \psi^{(k)}(s)\equiv 0,$$

$$s>t_0+T, \qquad k=0,1,\dots,n-1.$$

Theorem 5. If the relatively controllable system (1) is asymptotically stable for $u\equiv 0$, then for every initial state z_0 of system (1) there exists an optimal control.

4. Uniqueness of the optimal control.

Theorem 6. If the defining equation of system (1) is nondegenerate for a given m, then for every z_0 the optimal control is uniquely determined by virtue of (4) at least on the interval $[t_0, t_0+T^0-(m-1)h]$, $T^0\geq mh$.

Theorem 7. The optimal control for every state z_0 of a normal system (1) is unique.

5. On the correctness of the formulation of the time-optimal control problem. We shall say that the time-optimal control problem for (1) is formulated **relatively correctly** if the time T^0 depends continuously on the initial vector x_0

$$(T^0(x_0^k,\varphi(s))\to T^0(x^0,\varphi(s)) \quad \text{as } \|x_0^k-x_0\|\to 0).$$

The optimal control problem is formulated **correctly** if T^0 depends continuously on the initial state z_0 ($T^0(z_0^k)\to T^0(z_0)$ as $\|z_0^k-z_0\|\to 0$).

Theorem 8. If system (1) is normal and $B\varphi(s)\equiv\beta(s)c$, $t_0-h\leq s\leq t_0$, $|\beta(s)|\leq 1$, then the time-optimal control problem for (1) is formulated relatively correctly.

Theorem 9. The time-optimal control problem in the normal system (1) is formulated correctly.

6. On concepts connected with controllability. Following known works, one may introduce the notions of (relative) observability, control invariance, observation invariance, control autonomy, and observation autonomy in system (1). It is not difficult to determine the conditions under which the system possesses one or another of the listed properties. Since the technique for obtaining such conditions is standard, in the present note these conditions are omitted.

Nonstationary systems.

7. Constant delay. Let the motion of the point $x(t)$ in X be defined by the equation

$$\dot{x}(t)=A(t)x(t)+B(t)x(t-h)+c(t)u(t), \qquad t\in I_T, \tag{5}$$

where the elements of the matrices $A(t)$, $B(t)$, and of the vector $c(t)$ are defined and continuous on I_T together with $n-1$ derivatives.

The defining equation for (5) has the form

$$q_k(l,t) = A(t)q_{k-1}(l,t) + B(t)q_{k-1}(l-1,t-h) - \dot q_{k-1}(l,t), \tag{6}$$

$$q_1(1,t) \equiv c(t), \qquad q_0(l,t) \equiv 0, \qquad q_k(0,t) \equiv 0,$$

$$l = 1,2,\dots; \qquad k = 1,2,\dots; \qquad t \in I_T.$$

Equation (6) will be called **nondegenerate on I_T for a given** m, if the sequence $Q_m(T)$, $Q_m(T) = \{q_k(l,T),\ k = 1,2,\dots,n;\ l = 1,2,\dots,m\}$, has rank n.

The function $F(t,\tau)$, which determines c_T and S_{Tu} (see item 1), satisfies the equation

$$\frac{\partial F(t,\tau)}{\partial \tau} = -F(t,\tau)A(\tau) - F(t,\tau+h)B(\tau+h), \qquad F(t,t) = E,$$

$$F(t,\tau) \equiv 0, \qquad \tau > t,$$

$$c_T = Pz_0 = F(t_0+T,t_0)x_0 + \int_{t_0-h}^{t_0} F(t_0+T,\tau+h)B(\tau+h)\varphi(\tau)\,d\tau,$$

$$S_{Tu} = \int_{t_0}^{t_0+T} F(t_0+T,\tau)c(\tau)u(\tau)\,d\tau.$$

Theorems 1 (sufficiency), 2, 3, 4 carry over to system (5) without change. In Theorem 5 one must introduce the additional condition

$$\inf_{t_0 \ge 0} \varepsilon(t_0) \ge \gamma > 0.$$

8. Variable delay. The results of item 7, taking into account the remarks below, carry over to the system

$$\dot x(t) = A(t)x(t) + B(t)x(t-h(t)) + c(t)u(t), \qquad t \in I_T. \tag{7}$$

Here the scalar function $h(t)$, $h(t) > 0$, is continuously differentiable on I_T; $v = t - h(t)$ increases monotonically on I_T.

Let $t = r(v)$ be the inverse function for v. To equation (7) there corresponds the defining equation

$$q_k(l, r(t)) = \left[A(r(t))q_{k-1}(l, r(t)) + B(r(t))q_{k-1}(l-1, t) - \left.\frac{dq_{k-1}(l, s)}{ds}\right|_{s=r(t)} \right] \frac{dr(t)}{dr}, \tag{8}$$

$$q_1(1, t) \equiv c(t), \quad q_0(l, t) \equiv 0, \quad q_k(0, t) \equiv 0, \quad l = 1, 2, \dots; \ k = 1, 2, \dots, \ t \in I_T.$$

The quantities c_T, S_{Tu} for equation (7) are determined by the formulas

$$c_T = Pz_0 = F(t_0 + T, t_0)x_0 + \int_{t_0-h}^{t_0} F(t_0 + T, r(\tau))B(r(\tau))\varphi(\tau)\frac{dr(\tau)}{d\tau}\, d\tau,$$

$$S_{Tu} = \int_{t_0}^{t_0+T} F(t_0 + T, \tau)c(\tau)u(\tau)\, d\tau,$$

where

$$\frac{\partial F(t, \tau)}{\partial \tau} = -F(t, \tau)A(\tau) - F(t, r(\tau))B(r(\tau))\frac{dr(\tau)}{d\tau}, \qquad F(t, t) = E,$$

$$F(t, \tau) \equiv 0, \qquad \tau > t.$$

Remark 3. Above, the time-optimal problem under constraints (2) was considered. The generalization of the results obtained to other problems with constraints different from (2) is carried out according to known schemes ([3],[5]).

Ural Polytechnic Institute
named after S. M. Kirov

Received
26 IV 1966

References Cited

[1] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, *Mathematical Theory of Optimal Processes*, Moscow, 1961.
[2] N. Akhiezer, M. Krein, *On Certain Questions in the Theory of Moments*, Kharkov, 1938.
[3] R. Gabasov, F. M. Kirillova, *Avtomatika i telemekh.*, **25**, No. 7 (1964).

[4] F. M. Kirillova, *Izv. vyssh. uchebn. zaved., Matematika*, No. 4 (5) (1958).
[5] R. Gabasov, F. M. Kirillova, *DAN*, **156**, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.