SOME THEOREMS CONCERNING THE DIMENSION Ind

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Abstract

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SOME THEOREMS CONCERNING THE DIMENSION Ind

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Dauker ([1]) proves sum theorems for Ind * for hereditarily normal spaces. In this note we free ourselves from the condition of hereditary normality ** of the spaces participating in the sum, but impose conditions on the intersections of these spaces.

Theorem 1. *If a normal space $P = X \cup Y$ is the sum of two closed subsets X and Y such that: 1) $\operatorname{Ind} X \leq n$, $\operatorname{Ind} Y \leq n$; 2) $\operatorname{Ind}(X \cap Y) \leq 0$, then $\operatorname{Ind} P \leq n$.*

Proof. Let $F = X \cap Y$. Take a closed set Φ in P and an arbitrary neighborhood $O\Phi$ of it.

a) If $\Phi \cap F = \emptyset$, then for $O\Phi$ there will be found a neighborhood $O_1\Phi$ of the set Φ such that $[O_1\Phi] \cap F = \emptyset$, $O_1\Phi \subseteq O\Phi$, and $\operatorname{Ind} \operatorname{fr} O_1\Phi \leq n - 1$.

b) Let now $\Phi \cap F \neq \emptyset$. Put $\Phi_1 = \Phi \cap F$, $O_F\Phi_1 = O\Phi \cap F$, where the subscript F indicates in which set this $O_F\Phi_1$ is open.

Since $\operatorname{Ind} F \leq 0$, there will be found an open-and-closed neighborhood $O_F^1\Phi_1$ of the set Φ_1, contained in $O_F\Phi_1$.

Consider the sets $\Phi_2 = \Phi \cup O_F^1\Phi_1$ and $\Phi_2' = (P \setminus O\Phi) \cup F_1$, where $F_1 = F \setminus O_F^1\Phi_1$. The sets Φ_2 and Φ_2' are closed in P, and $\Phi_2 \cap \Phi_2' = \emptyset$. Next take the sets $X_1 = X \cap \Phi_2$, $X_1' = X \cap \Phi_2'$ and the sets $Y_1 = Y \cap \Phi_2$, $Y_1' = Y \cap \Phi_2'$. The sets X_1, X_1' (Y_1, Y_1') are closed in X (Y) and $X_1' \cap X_1 = \emptyset$ $(Y_1 \cap Y_2' = \emptyset)$; therefore, in view of the fact that $\operatorname{Ind} X \leq n$, $\operatorname{Ind} Y \leq n$, there will be found a neighborhood Γ_1 (Γ_2) of the set X_1 (Y_1) in X (Y) such that $\operatorname{Ind} \operatorname{fr} \Gamma_1 \leq n-1$ $(\operatorname{Ind} \operatorname{fr} \Gamma_2 \leq n-1)$ and $[\Gamma_1]_X \cap X_1' = \emptyset$ $([\Gamma_2]_Y \cap Y_1' = \emptyset)$. From the choice of the sets Γ_1 and Γ_2 it is clear that the set $\Gamma = \Gamma_1 \cup \Gamma_2$ is open in P, since $\Gamma_1 \cap Y = \Gamma_2 \cap X$. Moreover, $\operatorname{fr}_P \Gamma_1 \cap \operatorname{fr}_P \Gamma_2 = \emptyset$, since $\operatorname{fr}_P \Gamma_1 \subset X$, $\operatorname{fr}_P \Gamma_2 \subset Y$ and $\operatorname{fr}_P \Gamma_1 \cap F = \operatorname{fr}_P \Gamma_2 \cap F = \emptyset$, while $\operatorname{fr}_P \Gamma_1 \cup \operatorname{fr}_P \Gamma_2$ is closed. Then $\operatorname{Ind} \operatorname{fr}_P \Gamma \leq n - 1$ and $\Phi \subset \Gamma \subset O\Phi$. Thus theorem 1 is proved.

From theorem 1 it follows

Theorem 2. *Let a normal space*

$$P = \bigcup_{i=1}^{n} X_i,$$

where: 1) X_i *is closed in* P*; 2)* $\operatorname{Ind}(X_i \cap X_j) \leq 0$ *for* $i \neq j$ $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, n)$*; 3)* $\operatorname{Ind} X_i \leq n$ $(i = 1, 2, \dots, n)$*. Then* $\operatorname{Ind} P \leq n$*.*

Using theorem 2, we shall now prove theorem 3.

Theorem 3. *Let* $P = X \times Y$*, where* X, Y *are bicompacts and* $\operatorname{Ind} X \leq 1$*,* $\operatorname{Ind} Y \leq 1$*. Then* $\operatorname{Ind} P \leq 2$*.*

We shall first prove a lemma.

Lemma 1. *Let* X *be a one-dimensional bicompact,* $\operatorname{Ind} X \leq 1$*, and let* $\omega = \{O_1, O_2, \dots, O_s\}$ *be an arbitrary open covering. Then one can construct a refinement* ** $\alpha = \{A_1, A_2, \dots, A_s\}$, inscribed in ω, satisfying the condition $\operatorname{Ind} \operatorname{fr}(\operatorname{Int} A_j) \leq 0$.*

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* A space X has $\operatorname{Ind} X \leq n$ if for every closed set F and every neighborhood OF of it there exists $O_1 F \subset OF$ such that $\operatorname{Ind} \operatorname{fr} O_1 F \leq n - 1$, $\operatorname{Ind} \emptyset = -1$.

** All spaces considered in the note are assumed to be Hausdorff and normal.

*** A covering $\alpha = \{A_1, A_2, \dots, A_s\}$ of a space X is called a *partition* if $A_i = [\operatorname{Int} A_i]$ and $\operatorname{Int} A_i \cap \operatorname{Int} A_j = \emptyset$ for $i \neq j$ $(i, j = 1, 2, \dots, n)$.

Proof. Take an arbitrary open cover $\omega = \{O_1, O_2, \dots, O_s\}$. In ω inscribe such an open cover $\omega^1 = \{O_1^1, O_2^1, \dots, O_s^1\}$ that $[O_i^1] \subseteq O_i$ and $\operatorname{Ind} \operatorname{Fr} O_i^1 \leq 0$, $i = 1, 2, \dots, s$.

Define by induction the sets A_i $(i = 1, 2, \dots, s)$,

$$\operatorname{Int} A_1 = O_1^1 \setminus \bigcup_{i=2}^{s} [O_i^1], \qquad A_1 = [\operatorname{Int} A_1],$$

then

$$A_1 \cup \bigcup_{i=2}^{s} O_i^1 = X \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr}(\operatorname{Int} A_1) \leq 0,$$

since

$$\operatorname{Fr}(\operatorname{Int} A_1) \subseteq \bigcup_{i=1}^{s} \operatorname{Fr} O_i^1.$$

Suppose that we have already constructed sets $A_1, A_2, \dots, A_{k-1}$ satisfying the conditions:

a) $\quad \operatorname{Int} A_{i_1} \cap \operatorname{Int} A_{i_2} = \emptyset \quad$ for $i_1 \neq i_2$ $(i_1, i_2 = 1, 2, \dots, k-1)$;

$$\text{b)} \qquad \bigcup_{j=1}^{k-1} A_j \cup \bigcup_{i=k}^{s} O_i^1 = X;$$

$$\text{c)} \qquad \operatorname{Fr} \operatorname{Int} A_j \subseteq \bigcup_{i=1}^{s} \operatorname{Fr} O_i^1 \quad (j = 1, 2, \dots, k-1).$$

Construct A_k:

$$\operatorname{Int} A_k = O_k^1 \setminus \left(\bigcup_{i=k+1}^{s} [O_i^1] \cup \bigcup_{j=1}^{k-1} A_j \right), \qquad A_k = [\operatorname{Int} A_k].$$

Then, by the definition of the set A_k, we obtain:

$$\text{a)} \qquad \bigcup_{j=1}^{k} A_j \cup \bigcup_{i=k+1}^{s} O_i^1 = X.$$

Further,

$$\operatorname{Fr} \operatorname{Int} A_k \subseteq \bigcup_{i=k}^{s} \operatorname{Fr} O_i^1 \cup \bigcup_{i=1}^{k-1} \operatorname{Fr} \operatorname{Int} A_j,$$

but

$$\operatorname{Fr} \operatorname{Int} A_j \subseteq \bigcup_{i=1}^{s} \operatorname{Fr} O_i^1 \quad (j = 1, 2, \dots, k-1),$$

therefore

$$\text{b)} \qquad \operatorname{Fr} \operatorname{Int} A_k \subseteq \bigcup_{i=1}^{s} \operatorname{Fr} O_i^1$$

and

$$\text{c)} \qquad \operatorname{Int} A_k \cap \operatorname{Int} A_j = \emptyset \quad (j = 1, 2, \dots, k-1).$$

The partition $\alpha = \{A_1, A_2, \dots, A_s\}$ is the desired one. Indeed, α is inscribed in ω, and

$$\operatorname{Ind} \operatorname{Fr}(\operatorname{Int} A_j) \leq \operatorname{Ind} \left(\bigcup_{i=1}^{s} \operatorname{Fr} O_i^1 \right) \leq 0,$$

since $\operatorname{Ind} \operatorname{Fr} O_i^1 \leq 0$.

Proof of Theorem 3. Let $P = X \times Y$, where X, Y are bicompacta and $\operatorname{Ind} X \leq 1$, $\operatorname{Ind} Y \leq 1$. Let a closed set $F \subset P$ and its neighborhood OF be arbitrary. One may take a cover $\Omega = \{\omega_1 \times \omega_2\}$, whose elements are the sets $\{O_i \times G_j\}$ $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$, where $\omega_1 = \{O_1, \dots, O_n\}$ is an open cover of the bicompactum X, and $\omega_2 = \{G_1, \dots, G_m\}$ is an open cover of the bicompactum Y. In this case Ω can be chosen so that the following conditions will be fulfilled:

$$1) \quad \text{if } (O_i \times G_j) \cap F = \emptyset, \text{ then } [(O_i \times G_j)] \cap F = \emptyset;$$

$$2) \qquad \left[\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(O_j \times G_j)\right] \subseteq OF, \quad \text{where } [O_i \times G_j] \cap F \neq \emptyset.$$

Now, using Lemma 1, construct such partitions

$$\alpha = \{A_1, A_2, \dots, A_n\} \quad \text{and} \quad \beta_1 = \{B_1, B_2, \dots, B_m\}$$

of the spaces X and Y, respectively, that: a) α_1 is inscribed in ω_1, β is inscribed in ω_2; b) $\operatorname{Ind}\operatorname{Fr}(\operatorname{Int} A_i) \leq 0$ and $\operatorname{Ind}\operatorname{Fr}(\operatorname{Int} B_j) \leq 0$ $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$.

The partition

$$\alpha = \{\alpha_1 \times \alpha_2\},$$

consisting of the sets

$$A_i \times B_j \quad (i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$$

of the bicompactum P, has the property that

$$G = P \setminus \bigcup A_i \times B_j$$

(where $(A_i \times B_j) \cap F = \emptyset$) is open in P and $G \subseteq OF$. The boundary of G consists of closed subsets of sets of the form

$$\operatorname{Fr} A_i \times B_j \quad \text{and} \quad A_i \times \operatorname{Fr} B_j \quad (i = 1, 2, \dots, n;\ j = 1, 2, \dots, m),$$

and moreover these sets are such that if two of them intersect, then they intersect in a zero-dimensional closed set. Therefore, from Theorem 2 it follows that

$$\operatorname{Ind}\operatorname{Fr} G \leq 1.$$

Theorem 3 is proved.

Remark 1. Note that Lemma 1 is true for any normal space X, provided only that $\operatorname{Ind} X \leq 1$.

We now introduce two definitions.

Definition 1. We shall say that a closed set F is **normally situated** in a normal space X, if for every closed set $\Phi \subseteq F$ the subspace $X \setminus \Phi$ is normal.

Example. Let S be the bicompactum of O. V. Lokutsievskii ([2]). In this bicompactum the set $I \times \omega_1$ has the property that $S \setminus (I \times \omega_1)$ is normal, but the set $I \times \omega_1$ does not lie normally in the bicompactum S. If, however, a closed set $F \subseteq S$ is such that $F \cap (I \times \omega_1) = \emptyset$, then F lies normally in S.

Definition 2. A closed set F will be called **normally Ind-lying** in a normal space X if, for every closed set $\Phi \subseteq F$, the subspace $X \setminus F$ is normal and

$$\operatorname{Ind}(X \setminus \Phi) \leq \operatorname{Ind} X.$$

Theorem 4. *If a normal space P is the sum of two closed subsets X and Y in it such that: 1) $\operatorname{Ind} X \leq n$, $\operatorname{Ind} Y \leq n$; 2) $F = X \cap Y$ is a set normally Ind-lying in X and in Y; 3) $\operatorname{Ind} F \leq n$, then $\operatorname{Ind} P \leq n$.*

Before proving Theorem 4, we prove Theorem 5, from which Theorem 4 will easily follow.

Theorem 5. *Let P be a normal space. If in P there exists a closed set F such that $\operatorname{Ind} F \leq n$, $\operatorname{Ind}(P \setminus F) \leq n$, and F lies normally in P, then $\operatorname{Ind} P \leq n$.*

Proof. We shall prove the theorem by induction on the number n. For the sum of empty sets this is obvious. Suppose that Theorem 5 has already been proved for $n-1$, i.e. if P is a normal space and in P there exists a closed set F, normally lying in P, such that $\operatorname{Ind} F \leq n-1$ and $\operatorname{Ind}(P \setminus F) \leq n-1$, then $\operatorname{Ind} P \leq n-1$. We now prove Theorem 5 for n.

Let Φ be closed in P, and let $O\Phi$ be an arbitrary neighborhood of it.

a) If $\Phi \cap F = \emptyset$, then there exists a neighborhood $O_1\Phi \subseteq O\Phi$ such that

$$\operatorname{Ind} \operatorname{Fr} O_1\Phi \leq n-1.$$

b) Let $\Phi \cap F \neq \emptyset$. Denote by Φ_1 and $O\Phi_1$ the sets

$$\Phi_1 = \Phi \cap F, \qquad O\Phi_1 = O\Phi \cap F.$$

Since $\operatorname{Ind} F \leq n$, choose a neighborhood $O_1\Phi_1 \subseteq O\Phi_1$ of the set Φ_1 such that

$$\operatorname{Ind} \operatorname{Fr}_F O_1\Phi_1 \leq n-1.$$

Put

$$\Gamma = F \setminus [O_1\Phi_1], \qquad G_1 = P \setminus \operatorname{Fr}_F O_1\Phi_1.$$

Then G_1 is a normal space, Γ and $O_1\Phi_1$ are closed in G_1, and $\Gamma \cap O_1\Phi_1 = \emptyset$, $\operatorname{Ind}(P \setminus F) \leq n$. Consider the sets

$$\Phi_2 = \Phi \cup O_1\Phi_1, \qquad \Phi_2' = \Gamma \cup (G_1 \setminus O\Phi).$$

We shall have $\Phi_2 \cap \Phi_2' = \emptyset$, and Φ_2, Φ_2' are closed in G_1. Now choose open sets G_2 and G_2' in G_1 such that

$$\Phi_2 \subseteq G_2, \qquad \Phi_2' \subseteq G_2', \qquad [G_2]_{G_1} \cap [G_2']_{G_1} = \emptyset.$$

Next consider the sets

$$G_3 = G_2 \setminus F, \qquad G_3' = G_2' \setminus F.$$

We have

$$[G_3]_{(P\setminus F)} \cap [G_3']_{(P\setminus F)} = \emptyset.$$

Since $\operatorname{Ind}(P \setminus F) \leq n$, choose an open set G_4 in $P \setminus F$ such that

$$[G_3]_{(P\setminus F)} \subseteq G_4, \qquad [G_4]_{(P\setminus F)} \cap [G_3']_{(P\setminus F)} = \emptyset$$

and

$$\operatorname{Ind}\operatorname{Fr}_{(P\setminus F)} G_4 \leq n-1.$$

The set

$$G_5 = G_4 \cup O_1\Phi_1$$

is open in P and

$$\Phi \subseteq G_5 \subseteq O\Phi.$$

Notice that

$$\operatorname{Fr}_P G_5 = \operatorname{Fr}_{(P\setminus F)} G_4 \cup \operatorname{Fr}_F O_1\Phi_1,$$

where

$$\operatorname{Ind}\operatorname{Fr}_F O_1\Phi_1 \leq n-1,$$

$\operatorname{Fr}_F O_1\Phi_1$ lies normally in $\operatorname{Fr}_P G_5$, and

$$\operatorname{Ind}(\operatorname{Fr}_P G_5 \setminus \operatorname{Fr}_F O_1\Phi_1) = \operatorname{Ind}\operatorname{Fr}_{(P\setminus F)} G_4 \leq n-1.$$

Thus, $\operatorname{Ind}\operatorname{Fr}_P G_5 \leq n-1$. Theorem 5 is proved.

Remark 2. The bicompactum of Lokutsievskii ([2]) S is represented as the sum of the closed set $F = I\times\omega_1$ and the normal set $(S\setminus F)$, $\operatorname{Ind} F \leq 1$, $\operatorname{Ind}(S\setminus F) \leq 1$, while $\operatorname{Ind} S = 2$. But, as was said in Remark 1, the set F does not lie normally in S. Thus, the concept of a normally lying set is essential.

We now prove Theorem 4. Let

$$P = X \cup Y, \qquad F = X \cap Y.$$

Let a closed set $\Phi \subseteq P$ and its neighborhood $O\Phi$ be arbitrary.

a) If $\Phi \cap F = \emptyset$, then there exists a neighborhood $O_1\Phi \subseteq O\Phi$ such that

$$\operatorname{Ind}\operatorname{Fr} O_1\Phi \leq n-1.$$

b) Let

$$\Phi \cap F = \Phi_1 \neq \emptyset.$$

Put

$$O\Phi_1 = F \cap O\Phi.$$

Choose $O_1\Phi_1 \subseteq O\Phi_1$ in such a way that

$$\operatorname{Ind}\operatorname{Fr}_F O_1\Phi_1 \leq n-1.$$

Let

$$F_1 = F \setminus [O_1\Phi_1].$$

Then F_1 and $O_1\Phi_1$ are closed in

$$G = P \setminus \operatorname{Fr}_F O_1\Phi_1$$

and

$$F_1 \cap O_1\Phi_1 = \emptyset.$$

Consider now the sets

$$\Phi_2 = \Phi \cup O_1\Phi_1, \qquad \Phi_2' = F_2 \cup (G \setminus O\Phi).$$

The sets Φ_2, Φ_2' are closed in G and

$$\Phi_2 \cap \Phi_2' = \emptyset.$$

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now take the sets $X_1 = X \cap \Phi_2$, $X_1' = X \cap \Phi_2'$ and $Y_1 = Y \cap \Phi_2$, $Y_1' = Y \cap \Phi_2'$, $G_X = X \cap G$, $G_Y = Y \cap G$. We have $G_X = X \setminus \operatorname{bd}_F O_1\Phi_1$, $G_Y = Y \setminus \operatorname{bd}_F O_1\Phi_1$; consequently, G_X, G_Y are normal and $\operatorname{Ind} G_X \le n$, $\operatorname{Ind} G_Y \le n$. Therefore we can take an open subset Γ_X (Γ_Y) in G_X (G_Y) such that $X_1 \subset \Gamma_X$ ($Y_1 \subset \Gamma_Y$), $[\Gamma_X] \cap X_1' = \emptyset$, ($[\Gamma_Y] \cap Y_1' = \emptyset$), and $\operatorname{Ind} \operatorname{bd}_{G_X} \Gamma_X \le n-1$ ($\operatorname{Ind} \operatorname{bd}_{G_Y} \Gamma_Y \le n-1$). The set $\Gamma = \Gamma_X \cup \Gamma_Y$ is open in G, and hence open also in P, and

$$\operatorname{bd}_P \Gamma = \operatorname{bd}_{G_X} \Gamma_X \cup \operatorname{bd}_{G_Y} \Gamma_Y \cup \operatorname{bd}_F O_1\Phi_1,$$

where

$$\operatorname{bd}_{G_X} \Gamma_X \cup \operatorname{bd}_{G_Y} \Gamma_Y = \emptyset,$$

therefore

$$\operatorname{Ind}(\operatorname{bd}_{G_X} \Gamma_X \cap \operatorname{bd}_{G_Y} \Gamma_Y) \le n-1$$

and

$$\operatorname{Ind} \operatorname{bd}_F O_1\Phi_1 \le n-1.$$

Further, we see that $\operatorname{bd}_F O_1\Phi_1$ lies normally in $\operatorname{bd}_P \Gamma$ and

$$\operatorname{bd}_F O_1\Phi_1 \cap (\operatorname{bd}_{G_X} \Gamma_X \cup \operatorname{bd}_{G_Y} \Gamma_Y) = \emptyset;$$

consequently, by Theorem 5,

$$\operatorname{Ind} \operatorname{bd}_P \Gamma \le n-1.$$

Theorem 4 is proved. From Theorems 4 and 5 it follows that

Theorem $5'$ (Dowker ([1])). *Let P be a hereditarily normal space. If $F \subset P$ is closed and* $\operatorname{Ind} F \le n$, $\operatorname{Ind}(P \setminus F) \le n$, *then* $\operatorname{Ind} P \le n$.

Theorem $4'$**.** *Let $P = X \cup Y$ be a hereditarily normal space; X, Y are closed in P and* $\operatorname{Ind}(X \cap Y) \le n$, $\operatorname{Ind}(X \setminus F) \le n$, $\operatorname{Ind}(Y \setminus F) \le n$, *where $F = X \cap Y$. Then* $\operatorname{Ind} P \le n$.

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CITED LITERATURE

[1] C. H. Dowker, Quart. J. Math. Oxford, **4**, 267 (1953).
[2] O. V. Lokutsievskii, DAN, **67**, No. 2 (1949).

Note: Figure translations are in progress. See original paper for figures.