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Abstract

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ON THE STRUCTURE OF THE SET OF POINTS OF MUTUAL UNIQUENESS UNDER CONTINUOUS MAPPINGS OF MANIFOLDS

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Let f be a continuous mapping of an n-dimensional closed manifold M^n onto an n-dimensional closed manifold μ^n with degree c. We shall assume that the degree of the mapping is, in absolute value, greater than 1. A point $\xi \in \mu^n$ will be called a **point of mutual uniqueness** of the mapping f if its full inverse image consists of a single point. The set of all points of mutual uniqueness of the mapping f will be denoted by Ξ, and its full inverse image $f^{-1}(\Xi)$ by X.

In the present article the structure of this set is considered. The set of points of mutual uniqueness under mappings of n-dimensional manifolds was studied in the work of G. Hopf ([1]), in which the case where the mapping f is simplicial was examined in detail. It was shown that $\dim \Xi \leq n-2$, and the $(n-2)$-nd Betti number of the set Ξ was estimated in terms of the Betti numbers of the indicated manifolds. For $n = 2$ it was proved that, under continuous mappings, the set Ξ consists of a finite number of points, and this number was estimated in terms of the Euler characteristics of the manifolds M^n and μ^n.

Below the following propositions will be proved:

Theorem 1. The sets X and Ξ are homeomorphic.
Theorem 2. The set Ξ has type G_δ.
Theorem 3. The set Ξ does not separate any ball in μ^n.
Theorem 4. For $n = 3$, the set Ξ contains no locally connected continuum of index greater than two.

It follows from Theorem 1 that the assertions corresponding to Theorems 2–4 also hold for the set X.

Proof of Theorem 1. By the definition of the sets X and Ξ, the mapping $f : X \to \Xi$ is one-to-one and continuous. To establish the homeomorphy of these sets it is enough to establish the continuity of the mapping $f^{-1} : \Xi \to X$.

Let $\{\xi_n\}$ be a sequence of points of Ξ converging to the point $\xi_0 \in \Xi$. Consider the sequence $\{f^{-1}(\xi_n)\}$ in the set X, and let x be one of the cluster points of this sequence (such a point always exists, since M^n is compact). From the continuity of the mapping f it follows that $f(x) = \xi_0$, and since ξ_0 is a point of mutual uniqueness, $f^{-1}(\xi_0) = x$; consequently, x is the unique cluster point of the sequence $\{f^{-1}(\xi_n)\}$, and the mapping $f^{-1} : \Xi \to X$ is continuous at the point ξ_0, and, by the arbitrariness of the choice of ξ_0, everywhere on the set Ξ.

Proof of Theorem 2. First note that for any neighborhood $O(x)$ of a point $x \in M^n$ there is a neighborhood $O(\xi)$ of the point $\xi = f(x)$ possessing the property that $O(\xi)$ does not intersect the set $f(M^n \setminus O(x))$. Indeed, otherwise there would exist a sequence $\{x_n\}$ such that $x_n \in M^n \setminus O(x)$ and $\rho(f(x_n), \xi) \to 0$ as $n \to \infty$. Then, from the continuity of the mapping f, it follows that for any limit point x_0 of the sequence $\{x_n\}$ the relation $f(x_0) = \xi$ holds, which is impossible, since $f^{-1}(\xi)$ cannot coincide with the point x_0.

We shall now construct, for each point $x \in X$, a sequence of its neighborhoods $\{O(x, 1/n)\}$, where $O(x, 1/n) = \{y \mid \rho(x, y) < 1/n\}$. Let $O_n(\xi)$ be a neighborhood of the point $\xi = f(x)$, chosen as stated for the neighborhood $O(x, 1/n)$. Then

$$A_n = \bigcup_{\xi \in \Xi} O_n(\xi)$$

is an open set containing the set Ξ. Obviously,

$$\bigcap_{n=1}^{\infty} A_n \supset \Xi.$$

We shall show that

$$\bigcap_{n=1}^{\infty} A_n \subset \Xi.$$

Let $\eta \in \mu^n \setminus \Xi$; then the diameter of the full preimage of this point is nonzero; denote it by $d(\eta)$. None of the sets $O(x, 1/n)$, for $2/n < d(\eta)$, can contain the full preimage of the point η; consequently, by construction, the point η is not contained in any of the A_n for $n > 2/d(\eta)$, which proves our assertion. Thus,

$$\Xi = \bigcap_{n=1}^{\infty} A_n,$$

where all A_n are open in μ^n, and, consequently, it is of type G_δ.

In the proof of Theorems 1 and 2 we nowhere used the circumstance that μ^n and M^n are manifolds, and we imposed no restrictions on the degree of the mapping f. Therefore the indicated theorems are also valid for discontinuous mappings of metric spaces.

Proof of Theorem 3. We shall argue by contradiction. Suppose the set Ξ separates some ball T^n in the manifold μ^n. Then, as is known, there exists a closed set $\Psi \subset \Xi$ that also separates this ball. Denote by A_1 and A_2 the open subsets of T^n into which the set Ψ separates T^n. In the set $K = \Psi \cap T^n$, consider the subsets $K_1 = K \cap [A_1]$ and $K_2 = K \cap [A_2]$. Observe that $K_1 \cup K_2 = K$, for otherwise there would be a point $\xi \in K$ lying at a nonzero distance from both the set A_1 and the set A_2; in that case the set K, and hence also Ξ, would contain an n-dimensional ball, and the degree of the mapping f, contrary to the assumption, would be equal to $+1$ or -1.

Next, the set $K_1 \cap K_2$ is nonempty. Otherwise the sets $K_1 \cup A_1$ and $K_2 \cup A_2$ would be open, would not intersect, and together would make up the connected set T^n.

Now choose in the set $K_1 \cap K_2$ an arbitrary point η, and consider in the manifold M^n a ball neighborhood $O(x)$ of the point $x = f^{-1}(\eta)$ such that $f(O(x)) \subset T^n$. The set $f^{-1}(K) = W$ separates $O(x)$ into two open sets:

$$U = O(x) \cap f^{-1}(A_1)$$

and

$$V = O(x) \cap f^{-1}(A_2),$$

and, according to the choice of the point η, both sets U and V are nonempty.

Consider a discontinuous mapping $\varphi_0 : M^n \to S^n$, where S^n is the n-dimensional sphere, which contracts the set $M^n \setminus O(x)$ to a point $a \in S^n$ and maps $O(x)$ onto $S^n \setminus a$ homeomorphically. The degree of this mapping is obviously equal to $+1$ or -1.

We shall regard the sphere S^n as the unit sphere with center at the origin of the $(n+1)$-dimensional Euclidean coordinate space E^{n+1}; denote by S^{n-1} its intersection with the hyperplane $x_1 = 0$, and by E^+ and E^- the intersections of S^n with the half-spaces $x_1 > 0$ and $x_1 < 0$. We denote the points

$$(0, 0, \dots, 0, +1)$$

and

$$(0, 0, \dots, 0, -1)$$

by a^+ and a^-, respectively.

From results of the author [2] it follows that there exists a discontinuous deformation φ_t of the mapping φ_0 such that

$$\varphi_1((M^n \setminus O(x)) \cup W) = S^{n-1},$$

and, moreover,

$$\varphi_1(x) = a^+, \qquad \varphi_1(M^n \setminus O(x)) = a^-,$$

besides,

$$\varphi_1(U) = E^+, \qquad \varphi_1(V) = E^-.$$

Since the mappings φ_0 and φ_1 are homotopic, their degrees are equal.

The mapping

$$F = \varphi_1 f^{-1} : \Psi \to S^{n-1}$$

is discontinuous, since the mapping f^{-1} is discontinuous by Theorem 1, and the mapping φ_1 is so by construction. Extend the mapping F to the closed set $\Psi \cup (\mu^n \setminus T^n)$, putting

$$F(\alpha) = a^-$$

for all $\alpha \in \mu^n \setminus T^n$ that do not belong to Ψ. In the same work [2] it is shown that the mapping F can be extended to a mapping

$$F : \mu^n \to S^n$$

possessing the property that

$$F^{-1}(E^+) = A_1, \qquad F^{-1}(E^-) = A_2.$$

We denote the degree of the mapping F by C_1.

From the construction of the mappings φ_1 and Ff it follows that the points $\varphi_1(x)$ and $F(f(x))$ cannot be diametrically opposite on the sphere S^n for any x; consequently, by Hurewicz' s theorem ([3]), these mappings are homotopic and their degrees are equal. But the degree of the mapping Ff is equal to $C \cdot C_1$, where C and C_1 are integers and $|C| > 1$, while the degree of the mapping φ_1 is equal to $+1$ or -1. Thus a contradiction is obtained. Theorem 3 is proved.

From this result, however, one cannot draw a conclusion about the dimension of the set Ξ (the result conjectured here is $\dim \Xi \leq n-2$); moreover, there exists an example due to K. A. Sitnikov ([4]) of a two-dimensional set that does not divide any ball in E^3.

Proof of Theorem 4. We shall also prove this theorem by contradiction. Let K be a locally connected continuum belonging to Ξ and having at a point ξ index greater than two. Then, by the Menger–Nöbeling theorem ([5]), it contains at least three simple arcs having no common points except the point ξ. Without loss of generality one may assume that the indicated arcs are segments of length 1; we shall assume the same about their images. Choose in M^3 a sphere S^2 of sufficiently small radius $r < 1$, with center at the point $a = f^{-1}(\xi)$, such that the image of this sphere under the mapping f is at distance from the point ξ not greater than 1.

The sphere S^2 intersects each of the three mentioned segments in one point. Its image under the mapping f will have the same property with respect to the corresponding triple of segments in μ^3. Let Σ^2 be the sphere in μ^3 of radius 1 with center at the point ξ, and let h be the projection onto the sphere from its center. Then the mapping $hf : S^2 \to \Sigma^2$ is continuous and has degree C. At the three points—the ends of the chosen segments—this mapping is one-to-one. On the other hand, H. Hopf proved that the number of points of one-to-one correspondence under a continuous mapping of one two-dimensional sphere onto another with degree whose modulus is greater than 1 does not exceed two ([1]). We have obtained a contradiction. Theorem 4 is proved.

As was shown by P. S. Urysohn (6), every continuum of index not greater than two is a simple arc or a circle. Thus we obtain

Corollary to Theorem 4. *For $n = 3$, every locally connected continuum $K \subset \Xi$ is a simple arc or a circle.*

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Note: Figure translations are in progress. See original paper for figures.