ENGEL' S THEOREM FOR BINARY LIE ALGEBRAS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 519.48

MATHEMATICS

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ENGEL' S THEOREM FOR BINARY LIE ALGEBRAS

(Presented by Academician A. I. Mal' cev on 10 XII 1966)

The Engel theorem for finite-dimensional Lie algebras is well known; it establishes the nilpotency of a Lie algebra L under the condition that each of the operators adx, $x \in L$, is nilpotent. Here the nilpotency indices of the operators adx are, of course, bounded by the dimension of the algebra L. As Zorn showed (see ([1])), Engel' s theorem remains valid also in the case when the nilpotency indices of the operators adx are not assumed to be bounded collectively, while the finite-dimensionality condition on L is replaced by the maximality condition for subalgebras. Let us also note Kostrikin' s remarkable theorem on the local nilpotency of Lie algebras satisfying the n-th Engel condition ([2]). In a paper of A. I. Mal' cev ([3]) the class of binary Lie algebras was defined, as well as the class of ML-algebras, in a certain sense intermediate between Lie algebras and binary Lie algebras; subsequently ML-algebras came to be called Mal' cev algebras ([4]). For finite-dimensional Mal' cev algebras an analogue of Engel' s theorem turns out to be valid ([5]); we shall show that Engel' s theorem in Zorn' s formulation extends also to the broader class of binary Lie algebras.

Let A be a binary Lie algebra. Setting, for arbitrary $x, y, z \in A$,

$$J(x, y, z) = (xy)z + (yz)x + (zx)y,$$

we obtain that in A the identities

$$\begin{gathered} x^2 = 0, \\ J(xy, y, x) = 0, \end{gathered} \tag{1}$$

hold, or, taking (1) into account,

$$[(xy)y]x + [(yx)x]y = 0. \tag{2}$$

For convenience of notation, in expressions of the form $\{[(a_1a_2)a_3]\ldots\}a_k$ $(a_i \in A)$ we shall omit parentheses. Replacing y in (2) by $y + z$, we obtain the identity

$$xyzx + xzyx + yxxz + zxxy = 0. \tag{3}$$

Replacing x in (3) by $x + t$, we also obtain

$$xyzt + xzyt + yxtz + zxty + tyzx + tzyx + ytxz + ztxy = 0. \tag{4}$$

To each element $x \in A$ there corresponds the operator of right multiplication $R_x : a \to ax$, which is an endomorphism of the additive group A (in the case of Lie algebras this operator is denoted by adx). The operators R_x $(x \in A)$, with respect to the usual operations of addition, multiplication, and multiplication by scalars, generate an associative algebra $\bar{A}$, called the **algebra of right multiplications of the algebra** A. If B is a subalgebra in A, then by $\bar{B}$ we denote the subalgebra in $\bar{A}$ generated by all operators of the form R_x, $x \in B$. Relations (3), (4) mean that in $\bar{A}$ relations of the form

$$R_{xR_{yR}}x = R_x^2 R_y + R_{yx} R_x - R_{yxx}, \tag{3'}$$

$$R_{zR_{yR}}x = R_{xR_{zR}}y - R_{xR_{yR}}z + R_{zR_{xR}}y + R_{yx}R_z + R_{yz}R_x - R_{yxz} - R_{yzx}. \tag{4'}$$

One says that an algebra A **satisfies Engel' s condition** if, for every $x \in A$, the operator R_x is nilpotent.

Theorem. *If a binary Lie algebra A satisfies Engel' s condition and the maximality condition for subalgebras, then the algebra A is nilpotent.*

Proof. The assertion that the algebra A is nilpotent is obviously equivalent to the assertion that the algebra $\bar{A}$ is nilpotent. Suppose that the algebra A, and consequently the algebra $\bar{A}$, is not nilpotent. Consider the set σ of subalgebras $B \subseteq A$ for which the algebra $\bar{B}$ is nilpotent. The set σ is nonempty, since, for example, every one-dimensional subalgebra of the algebra A belongs to σ. Let B be a maximal element of σ, and let n be the index of nilpotency of $\bar{B}$. We shall show that in $A \setminus B$ there exists an element s such that

$$(*) \quad sb \in B \quad \text{for every } b \in B.$$

Indeed, take an arbitrary element $s_0 \in A \setminus B$. If $s = s_0$ does not satisfy condition $(*)$, then there exists an element $b_1 \in B$ such that $s_1 = s_0 b_1 \in A \setminus B$. Repeating this reasoning, we obtain a sequence of elements s_k $(s_k = s_{k-1} b_k)$ such that $s_k \in A \setminus B$. This process must terminate at some step, since $s_n = s_0 b_1 b_2 \dots b_n = s_0 R_{b_1 b_2 \dots R_{b_n}} = 0 \in B$. Denote by C the algebra $B + (s)$, $B \subset C \subseteq A$, and prove that the algebra $\bar{C}$ is nilpotent. The resulting contradiction to the maximality of the subalgebra B proves the theorem.

Let $\langle c_1 c_2 \dots c_t \rangle$ denote a product of the elements $c_1, c_2, \dots, c_t \in C$ with an arbitrary placement of parentheses, and let C^t be the subalgebra in C generated by all possible elements of the form $\langle c_1 c_2 \dots c_t \rangle$, $c_i \in C$. We shall call the **degree** of an element $c \in C$ the maximal number t such that $c \in C^t$ (in the case where this number exists). Denote this number by $\nu(c)$. If

$$c \in \bigcap_{t=1}^{\infty} C^t,$$

then set $\nu(c) = \infty$. Suppose some element $R \in \bar{C}$ is written in the form

$$R = R_{c_1} R_{c_2} \dots R_{c_k} \quad (c_i \in C).$$

Then put $\nu(R) = \sum_{i=1}^{k} \nu(c_i)$. The number $\nu(R)$ will also be called the **degree** of R. If $R = 0$, then we agree to take $\nu(R) = \infty$. We note that here the degree is assigned essentially not to the element $R \in \bar{C}$ itself, but to its written form.

Lemma 1. *Let each of the elements c_i $(1 \le i \le k,\ k \ge 1)$ either belong to B or be equal to s. Then the expression $R = R_{c_1} \dots R_{c_k}$ can be represented as a linear combination of analogous expressions R_j, with $\nu(R_j) \ge \nu(R)$ for every j, and R_j contains in its writing no subwords of the form $R_{zR_{yR}} s$ and $R_{zR_{yR}} s$, where $y, z \in B$.*

For the proof, relations $(3')$, $(4')$ and induction on the number $g(R)$ are used, where

$$g(R) = k + \sum_{i=1}^{k} p(c_i), \qquad p(c_i) = \begin{cases} i, & c_i = s; \\ 0, & c_i \in B. \end{cases}$$

Lemma 2. *Let $c = c_1 c_2 \dots c_m$ $(m \ge 1)$, where each of the elements c_i either belongs to B or is equal to s; $c_1 \in B$. Then for every $k \ge 1$ there exists a number $N(k)$ such that, for all $m \ge N(k)$, the element c is representable as a linear combination of analogous expressions (possibly of smaller length) in which the first k factors belong to B, and the first factor contains c_1. If r is the nilpotency exponent of the operator R_s, then one may take*

$$N(k) = (2^{k-1} - 1)r + 1.$$

For $k = 1, 2$ the lemma is obvious. Suppose the lemma has already been proved for $k-1$; then the element c is considered for $m \ge 2N(k-1) + r - 1$, and the induction step is carried out with the help of Lemma 1. Setting $N(1) = 1$, $N(k) = 2N(k-1) + r - 1$, we obtain $N(k) = (2^{k-1} - 1)r + 1$. From Lemma 2 it follows, obviously, that the algebra C is nilpotent; using Lemma 1, we conclude from this that the algebra $\bar{C}$ is also nilpotent. The theorem is proved.

Remark 1. The proof of the theorem remains valid also for arbitrary binary Lie rings satisfying the Engel condition and the maximality condition for subrings.

Remark 2. The scheme of the proof of Engel' s theorem proposed in the present paper also applies to some other classes of anticommutative algebras with fourth-degree identities, for example, to algebras defined by the identities

$$x^2 = 0,$$

$$xyzt - xy\,tz - zt\,xy + zt\,yx + \gamma(xy)(tz) = 0,$$

where γ is an arbitrary scalar.

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Received
29 XII 1966

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Note: Figure translations are in progress. See original paper for figures.