ON THE CONVERGENCE OF SINGULAR INTEGRALS AT LEBESGUE-ORLICZ POINTS OF FUNCTIONS OF SYMMETRIC SPACES

1967

SovietRxiv

Abstract

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MATHEMATICS

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ON THE CONVERGENCE OF SINGULAR INTEGRALS AT LEBESGUE-ORLICZ POINTS OF FUNCTIONS OF SYMMETRIC SPACES

(Presented by Academician M. A. Lavrent' ev, September 24, 1966)

Denote by E a Banach symmetric space of functions measurable on $[0;1]$ ([1]). The formula $\|\chi_e\|_E = \varphi(me)$, where $\chi_e(t)$ is the characteristic function of the measurable set $e \subset [0;1]$, defines the function $\varphi(t)$, called the fundamental function. We shall assume that the function $\varphi(t)$ is monotonically increasing, concave, and $\varphi(t) \to 0$ as $t \to 0$.

The present paper consists of two parts, naturally connected with each other. In the first part the notion of a Lebesgue-Orlicz point of a function from the symmetric space E is introduced, and the properties of the set of such points are described. In the second part the question of the representability of functions $x \in E$ by a singular integral at Lebesgue-Orlicz points is considered.

1. Denote by $\chi_h^{t_0}$ the characteristic function of the interval $(t_0 - h;\, t_0 + h)$.

Definition. We shall call a point $t \in [0;1]$ a **Lebesgue-Orlicz point** of the function $x \in E$ if

$$\lim_{h \to 0} \{\, \|[x(t) - x(t_0)]\chi_h^{t_0}\|_E / \|\chi_h^{t_0}\|_E \,\} = 0. \tag{1}$$

If $E = L_p$ $(p > 1)$, then definition (1) coincides with the definition of a Lebesgue point of order p. For $p = 1$, (1) means that the point t_0 is a Lebesgue point of the function $x(t)$. In the case when the space E is the Orlicz space L_M^*, definition (1) coincides with the definition of a Lebesgue-Orlicz point of a function $x \in L_M^*$, introduced in ([2]). This follows from the relation $u < M^{-1}(u)N^{-1}(u) \le 2u$, valid for all $u > 0$, and from the formula for the norm of a characteristic function in an Orlicz space.

We shall say that the fundamental function $\varphi(t)$ **satisfies condition** (α) if $\varphi[t\alpha(t)]/\varphi(t) \to 0$ as $t \to 0$ for any function $\alpha(t) \to 0$.

It is not difficult to prove the following theorem.

Theorem 1. *If the fundamental function $\varphi(t)$ satisfies condition (α), then the union of the set of density points of the set $e \subset [0;1]$ and the set of density points* *of the set* $\complement e$ *coincides with the set of Lebesgue–Orlicz points of the characteristic function* $\chi_e(t)$.

Theorem 2. *In order that every bounded function* $x \in E$ *have a set of Lebesgue–Orlicz points of full measure, it is necessary and sufficient that the fundamental function* $\varphi(t)$ *satisfy condition* (α).

We give the scheme of the proof of necessity. If condition (α) is not satisfied, then for some function $\alpha(t)$ there exists a sequence $\{h_n\}$ such that for every n one has $\alpha(h_n) < 1/(2^{n+1}+1)$ and $\varphi[h_n\alpha(h_n)] \geq \alpha_0\varphi(h_n)$, where $\alpha_0 > 0$. Let a function $\psi(h)$ be defined on $(0;1)$, monotonically increasing, and $\psi(h_n) = 1/2^{n+2}$. Construct an increasing sequence of integers $\{k_n\}$ such that, for every n,

$$(2^{-1}+2^{-n-1})/2^{k_n+1} < \psi^{-1}(2^{-n-2}) \leq (2^{-1}+2^{-n-1})/2^{k_n}.$$

From the sequence $\{k_n\}$ we construct in $[0;1]$ sets F and $\mathcal{E}$ by the method indicated in (3),

p. 73. Here $m\mathcal{E} = 1/2$, and every point of the set $\mathcal{E}$ is not a Lebesgue–Orlicz point of the function $\chi_F(t)$.

In the case where E is an Orlicz space, condition (α) is equivalent to the Δ_2-condition. Therefore, from Theorem 2 follows the first part of the main theorem in (3).

Theorem 2 is sharp in the following sense. There exists a symmetric space E, whose fundamental function satisfies condition (α), and which contains a function that is the limit of bounded functions, such that the measure of its set of points that are not Lebesgue–Orlicz points is positive. As an example of such a space one may take the Orlicz space L_M^*, generated by an N-function $M(u)$ satisfying the Δ_2-condition and not satisfying the Δ'-condition (4). Indeed, on the basis of Theorem 5 (3), in L_M^* there exists a function whose set of points not being Lebesgue–Orlicz points has positive measure. It remains to recall that the Δ_2-condition is equivalent to condition (α).

Suppose that the function $\varphi(t)$ does not satisfy condition (α). Let the function $\alpha(t)$ be such that

$$\lim_{t\to 0}\{\varphi[t\alpha(t)]/\varphi(t)\} = 0. \tag{2}$$

Let $e \subset [0;1]$, and let t_0 be a density point of the set e. We shall say that the set e has at the point t_0 a density order of type (α) if there exists an $h_0 > 0$ such that for all $h < h_0$ one has

$$[me \cap (t_0-h;t_0+h)]/2h \geq 1-\alpha(2h),$$

where $\alpha(t)$ satisfies (2).

Theorem 3. *In order that the point t_0 be a Lebesgue–Orlicz point of a bounded function $x \in E$, it is necessary and sufficient, and for an unbounded function it is necessary, that there exist a set $e \subset [0;1]$ such that:*

1. *The function $x(t)$ is continuous with respect to e at the point t_0.*
2. *The density order of the set e at the point t_0 is of type (α).*

If the function $x \in E$ is the limit of bounded functions, then

$$\lim_{me \to 0} \|x\chi_e\|_E = 0.$$

Put $\gamma(t) = \sup \|x\chi_e\|_E$, where the supremum is taken over all sets $e \subset [0;1]$ such that $me \le t$.

Theorem 4. *In order that the point t_0 be a Lebesgue–Orlicz point of a function $x \in E$, which is the limit of bounded functions, it is sufficient that there exist a set $e \subset [0;1]$ such that:*

1. *The function $x(t)$ is continuous with respect to e at the point t_0.*
2. *The density order of the set e at the point t_0 satisfies the condition*

$$[me \cap (t_0 - h; t_0 + h)]/2h \ge 1 - \gamma^{-1}[\alpha(2h)\varphi(2h)]/2h,$$

where $\alpha(t) \to 0$ as $t \to 0$, and the function $\gamma^{-1}(t)$ is the inverse of the function $\gamma(t)$.

§ 2. In the paper ([5]) B. I. Korenblyum formulated a necessary and sufficient condition for representability at Lebesgue points of a function $x \in L_p$ $(p > 1)$ by a singular integral:

$$x(t) = \lim_{n\to\infty} \int_0^1 k_n(s,t)x(s)\,ds. \tag{3}$$

In analogous form, C. Tandori ([6]) and R. Taberski ([7]) found necessary and sufficient conditions for representability by formula (3) of functions from the spaces L_p and Orlicz at Lebesgue–Orlicz points. Below Theorem 7 is presented, from which the results of R. Taberski follow. The basis of the proof of Theorem 7 is the scheme set forth by B. I. Korenblyum in ([5]).

Denote by E_0 the closure in the space E of the set of bounded functions. It is known that the general form of a functional in E_0 is integral. Denote by R_E the linear set of functions $x \in E_0$ such that,

$$\lim_{h\to 0}\{\|x\chi_h\|_E / \|\chi_h\|_E\} = 0, \tag{4}$$

where $\chi_h(t)$ is the characteristic function of the interval $(0;h)$.

Relation (4) means that the class R_E contains precisely those functions of the space E_0 which vanish at $t=0$ and for which the point $t=0$ is a Lebesgue-Orlicz point.

Let $x \in R_E$. Put

$$\|x\|_{R_E} = \sup_{0<h\leqslant 1} \{\|x\chi_h\|_E / \|\chi_h\|_E\}.$$

The space R_E is complete.

Theorem 5. *If the function $k(t)$ has the property that for every function $x \in R_E$ there exists*

$$I(x) = \lim_{h\to 0} \int_h^1 x(t)k(t)\,dt,$$

then I is a linear functional in R_E.

For the proof, note that the function $k(t)\chi_{(h,1)}(t)$ is a linear functional in E_0, and

$$|I_h(x)| \leqslant \varphi(1)\|x\|_{R_E}\|k\chi_{(h,1)}\|_{E_0^*},$$

where

$$I_h(x) = \int_h^1 x(t)k(t)\,d(t),$$

whence it follows that for every h, I_h is a linear functional in R_E. It remains to apply the Banach-Steinhaus theorem.

Denote by $\|I\|_R$ the norm of the functional I in R_E. Let $h_0 = 1$ and let $q < 1$. Denote by h_n $(n = 0, 1, 2, ...)$ the numbers defined by the formulas

$$h_{n+1} = \varphi^{-1}[q\varphi(h_n)], \tag{5}$$

where $\varphi^{-1}(t)$ is the function inverse to $\varphi(t)$. Since $\varphi(t)$ is concave, it follows from (5) that $h_n \leqslant q^n$. Denote by $C(k)$ the expression

$$C(k) = \sum_{n=0}^{\infty} \varphi(h_n)\|k\chi_{(h_{n+1},\,h_n)}\|_{E_0^*}.$$

Lemma. *The following inequality holds:*

$$q(1-q)C(k) \leqslant \|I\|_R \leqslant C(k). \tag{6}$$

We indicate the main points of the proof. Let the function $k(t)$ be such that $C(k) < \infty$. Let $x \in R_E$. Then

$$|I(x)| \leqslant \sum_{n=0}^{\infty} \left| \int_{h_{n+1}}^{h_n} x(t)k(t)\,dt \right| \leqslant$$

$$\leqslant \sum_{n=0}^{\infty} \|x\chi_{(h_{n+1},\,h_n)}\|_E \|k\chi_{(h_{n+1},\,h_n)}\|_{E_0^*} \leqslant \sum_{n=0}^{\infty} \varphi(h_n)\|x\|_{R_E}\|k\chi_{(h_{n+1},\,h_n)}\|_{E_0^*},$$

whence the right-hand side of inequality (6) follows. Let $\|I\|_R < \infty$. Then

$$\int_{h_{n+1}}^{h_n} x(t)k(t)\,dt < \infty$$

for every $x \in E_0$. Therefore $k(t)\chi_{(h_{n+1},\,h_n)}(t) \in E_0^*$ $(n = 0, 1, 2, \ldots)$. Let $x_n \in E_0$ be such functions that, for every n, $\|x_n\chi_{(h_{n+1},\,h_n)}\|_E = 1$ and

$$\left|\int_{h_{n+1}}^{h_n} x_n(t)\,k(t)\,dt\right| \geq \left\|k\chi_{(h_{n+1},\,h_n)}\right\|_{E_0^*} - \varepsilon.$$

Take the function

$$x_p^* = \begin{cases} \varphi(h_n)\,|x_n|\ \operatorname{sign} k(t), & t \in (h_{n+1}, h_n), \quad n = 0, 1, 2, \ldots, p, \\ 0, & t \in (0, h_{p+1}). \end{cases}$$

It turns out that

$$\|x_p^*\chi_h\|_E \leq \varphi(h)/q(1-q),$$

whence it follows that $x_p^* \in R_E$ and

$$\|x_p^*\|_{R_E} \leq 1/q(1-q)$$

for every p. Next we find

$$\|I\|_R\|x_p^*\|_{R_E} \geq \left|\int_0^1 x_p^*(t)\,k(t)\,dt\right| =$$

$$= \sum_{n=0}^{p} \varphi(h_n) \int_{h_{n+1}}^{h_n} |x_n(t)k(t)|\,dt \geq \sum_{n=0}^{p} \varphi(h_n) \left\|k\chi_{(h_{n+1},\,h_n)}\right\|_{E_0^*} - \varepsilon \sum_{n=0}^{p} \varphi(h_n),$$

from which the left-hand side of inequality (6) follows.

The following theorem follows from the lemma.

Theorem 6. *The general form of a functional in R_E can be given by the formula*

$$I(x) = \int_0^1 k(t)x(t)\,dt,$$

where $k(t)$ is such a function that, for every h, $k(t)\chi_{(h,1)}(t) \in E_0^*$, and the inequality $C(k) < \infty$ holds.

Denote by

$$I_n(x) = \int_0^1 x(t)k_n(t)\,dt \quad (n = 1, 2, \dots).$$

Theorem 6 allows us to obtain the following theorem.

Theorem 7. *For any function $x \in E_0$ for which the point $t = 0$ is a Lebesgue-Orlicz point, the relation*

$$\lim_{n\to\infty} I_n(x) = x(0)$$

holds if and only if:

1. $$\lim_{n\to\infty} \int_0^h k_n(t)\,dt = 1 \quad \text{for every } h, \quad 0 < h \leq 1.$$

2. $$\overline{\lim}_{n\to\infty} \sum_{\nu=0}^{\infty} \varphi(h_\nu) \left\| k_n \chi_{(h_{\nu+1},\, h_\nu)} \right\|_{E_0^*} < \infty,$$

where the numbers h_ν are defined by formulas (5); $q < 1$ and $h_0 = 1$.

The author expresses sincere gratitude to S. T. Krein for his attention.

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Received
24 VIII 1966

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