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PHYSICS

1967

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Abstract

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UDC 535.13

PHYSICS

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PROPAGATION OF WAVE BEAMS IN NON-LINEAR MEDIA

(Presented by Academician A. M. Prokhorov, 19 XI 1966)

Recently a number of works have appeared devoted to the theory of the propagation of wave beams in nonlinear media. In work ([1]) the possibility of self-focusing of beams in such media was noted. In subsequent works ([2,3]) beams of self-sustaining form were considered, and, finally, in works ([4–6]) the problem of the propagation of a beam in a medium with a Gaussian initial intensity distribution was considered. In works ([3,4]) the concept of the critical field was introduced. For the case when the initial field considerably exceeds the critical one, work ([4]) gives an expression for the characteristic length of field variation along the beam axis. Since the nonlinearity of the medium at attainable fields is weak, one usually restricts oneself only to the first term of the expansion of the dielectric permittivity in powers of $\mathcal{E}^2$ ([3]):

$$\varepsilon(\mathcal{E}^2) = \varepsilon_0 + \varepsilon_2\mathcal{E}^2 \quad (\varepsilon_0 > 0,\ \varepsilon_2 > 0). \tag{1}$$

The monochromatic field $\mathcal{E}$ (which below, for simplicity, we shall regard as scalar) can always be represented in the form

$$\mathcal{E} = \frac{1}{2}Ee^{i(kz-\omega t)} + \text{c.c.} \tag{2}$$

Maxwell's equations together with the material equation (1) lead to the equation for the amplitude $E(x, y, z)$:

$$\Delta E + 2ik\frac{\partial E}{\partial z} + k^2 n_2 |E|^2 E = 0, \tag{3}$$

where $k = \dfrac{\omega}{c}\sqrt{\varepsilon_0}$, $n_2 = \dfrac{\varepsilon_2}{2\varepsilon_0}$. Below we shall assume that the beam has axial symmetry (the beam axis coincides with the z-axis). Under ordinarily realized conditions the following relation is always satisfied: the characteristic length of variation of the field E along the beam axis is considerably greater than the

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corresponding length in the transverse direction. Under this condition one may neglect the term $\partial^2 E/\partial z^2$ in equation (3). As a result one obtains the equation used in most of the cited works,

$$\partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 + 2ik\,\partial E/\partial z + k^2 n_2 |E|^2 E = 0. \tag{4}$$

On the basis of equation (4), works [5,6] analyze the question of the propagation of wave beams in nonlinear media. However, this analysis is in fact based on a number of further assumptions. It is shown below that the further assumptions made in works [5,6] are not satisfied and that a correct analysis leads to a different picture of the phenomenon.

Let us represent the field E in the form

$$E = e^A, \tag{5}$$

where A is a new unknown function of the coordinates. Substituting this expression into equality (4), we find the equation for the quantity A:

$$\begin{gathered}\partial^2 A/\partial x^2 + \partial^2 A/\partial y^2 + (\partial A/\partial x)^2 \\ +(\partial A/\partial y)^2 + 2ik\,\partial A/\partial z + k^2 n_2 e^{2A^r} = 0\end{gathered} \tag{6}$$

($A^r = \operatorname{Re} A$). We shall assume that at $z = 0$ the field is a function of the quantity $q = x^2 + y^2$. In doing so we shall restrict ourselves to considering only analytic functions of x, y (for which the Taylor expansion must contain only integral nonnegative powers of q). Obviously, the field in the medium (for $z > 0$) will also be a function of q, and the quantity A can be represented in the form

$$A(q,z) = A_0(z) + qA_1(z) + q^2 A_2(z) + \cdots \tag{7}$$

Substituting equality (7) into equation (6), we arrive at a system of equations for the quantities $A_n(z)$

$$2ik\,\frac{dA_{n-1}}{dz} + 4\left\{n^2 A_n + \sum_{k=0}^{n} k(n-k)A_{kA_{n-k}}\right\} + k^2 n_2 e^{2A_0^r} L_{n-1} = 0$$

$$(n = 1, 2, ...), \tag{8}$$

where the real coefficients L_k are determined by the equality

$$e^{2(qA_1^r + q^2 A_2^r + ...)} = 1 + qL_1 + q^2 L_2 + \cdots \tag{9}$$

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Introducing the notation $a_0 = e^{2A_0^r}$, $b_0 = \frac{1}{k}\frac{dA_0^i}{dz}$ $(A^i = \operatorname{Im} A)$, $k^{-2m}A_m =$

$$= a_m + ib_m \quad (m = 1, 2, ...), \quad u = kz$$

and putting in equality (8) $n = 1, 2, 3, 4, 5$, we arrive at a system of equations for the quantities a_k, b_k:

$$\begin{aligned}
&a_0' + 4b_1a_0 = 0,\\
&a_1' + 4b_1a_1 = -8b_2,\\
&b_1' + 2b_1^2 = n_2a_0a_1 + 2a_1^2 + 8a_2,\\
&a_2' + 8b_1a_2 = -8a_1b_2 - 18b_3,\\
&b_2' + 8b_1b_2 = n_2a_0(a_1^2 + a_2) + 8a_1a_2 + 18a_3,\\
&a_3' + 12b_1a_3 = -4(3a_1b_3 + 4a_2b_2) - 32b_4,\\
&b_3' + 12b_1b_3 = n_2a_0\left(\tfrac{2}{3}a_1^3 + 2a_1a_2 + a_3\right) + 8(a_2^2 - b_2^2) + 12a_1a_3 + 32a_4,\\
&a_4' + 16b_1a_4 = -8(2a_1b_4 + 3a_2b_3 + 3a_3b_2) - 50b_5,\\
&\dots
\end{aligned} \tag{10}$$

The prime denotes differentiation with respect to u. For the quantity b_0 one then obtains the relation, not connected with system (10),

$$2b_0 = n_2a_0 + 4a_1, \tag{11}$$

which determines the correction to the longitudinal wave number on the axis of the beam.

Thus, for the coefficients a_k, b_k an infinite system of coupled equations is obtained. If this system is formally truncated, leaving only the first 3 equations and setting $a_2 \equiv b_2 \equiv 0$, then the equations obtained in this way coincide (up to a change of variables) with the initial equations adopted in works ([5,6]). We shall not restrict ourselves in advance to any definite number of equations of system (10). Suppose that at $z = 0$ the beam under consideration has a plane phase front on the axis ($b_1^{(0)} = 0$; $b_2^{(0)}, b_3^{(0)}, ...$ arbitrary) and an arbitrary intensity distribution (i.e. an arbitrary set of coefficients $a_0^{(0)}, a_1^{(0)}, a_2^{(0)}, ...$, with $a_0^{(0)} > 0$, $a_1^{(0)} < 0$). Then from the first three equations it follows at once that $a_0'(0) = 0$ and that $a_0''(0) > 0$ if $a_0^{(0)} > a_0^{\text{cr}}$; $a_0''(0) < 0$, if $a_0^{(0)} < a_0^{\text{cr}}$, where

$$n_2a_0^{\text{cr}} \equiv n_2E_{\text{cr}}^2 = -2a_1^{(0)} - 8\,\frac{a_2^{(0)}}{a_1^{(0)}}. \tag{12}$$

It follows from formula (12) that the expression for the critical field, besides the coefficient $a_1^{(0)}$, also contains the coefficient $a_2^{(0)}$, which was not taken into

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account in previous works. The presence of this coefficient explains, for example, the difference between the expressions for the critical field given in works ([3],[4]) (for a beam with a Gaussian initial distribution ([4]), $a_2^{(0)} = 0$; for a beam of self-similar form ([3]), $a_2^{(0)} \simeq {}^3/_4 (a_1^{(0)})^2$). It is also interesting to note that if the right-hand side of expression (12) is negative, then the axial field increases (in sufficient proximity to the boundary $z = 0$) even in the absence of nonlinearity of the medium. This increase is due only to the initial form of the beam and occurs at an arbitrarily small intensity.

Below, for simplicity, we shall assume that at $z = 0$ the beam has a plane phase front ($b_1^{(0)} = b_2^{(0)} = ... = 0$) and a Gaussian intensity distribution ($a_2^{(0)} = a_3^{(0)} = ... = 0$). Let us represent all the coefficients a_k, b_k in the form of Taylor series in powers of u. From equations (10) it follows immediately that in the expressions for a_k only the even terms of the expansions are different from zero, while in the expressions for b_k only the odd terms are:

$$\begin{aligned}
a_0 &= a_0^{(0)} + u^2 a_0^{(2)} + u^4 a_0^{(4)} + \dots, & b_1 &= u b_1^{(1)} + u^3 b_1^{(3)} + \dots, \\
a_1 &= a_1^{(0)} + u^2 a_1^{(2)} + u^4 a_1^{(4)} + \dots, & b_2 &= u b_2^{(1)} + u^3 b_2^{(3)} + \dots, \\
a_2 &= u^2 a_2^{(2)} + u^4 a_2^{(4)} + \dots, & b_3 &= u b_3^{(1)} + u^3 b_3^{(3)} + \dots, \\
&\dots\dots\dots\dots\dots & &\dots\dots\dots\dots
\end{aligned} \tag{13}$$

It is clear in advance that, applying the expansions (13), we shall obtain a solution of the problem only near the boundary of the medium $u = 0$. However, for small excesses of the initial field over the critical one, this turns out to be sufficient to reveal the essential aspects of the phenomenon.

Substituting the expansion (13) into equations (10), it is not difficult to obtain explicit expressions for the coefficients $b_3^{(1)}, b_2^{(1)}, b_1^{(1)}, b_1^{(3)}, a_2^{(2)}, a_1^{(2)}, a_0^{(2)}, a_0^{(4)}$. In doing so, the first 7 equations of system (10)* are needed. We give some of the expressions obtained:

$$\begin{aligned}
b_1^{(1)} &= n_2 a_0^{(0)} a_1^{(0)} + 2(a_1^{(0)})^2, \\
b_1^{(3)} &= -2(b_1^{(1)})^2 - \frac{4}{3} n_2 a_0^{(0)} a_1^{(0)} \left[b_1^{(1)} + 22(a_1^{(0)})^2 \right], \\
a_2^{(2)} &= -10 n_2 a_0^{(0)} (a_1^{(0)})^3, \\
a_1^{(2)} &= -2 a_1^{(0)} b_1^{(1)} - 4 n_2 a_0^{(0)} (a_1^{(0)})^2, \\
a_0^{(2)} &= -2 a_0^{(0)} b_1^{(1)}, \\
a_0^{(4)} &= 4 a_0^{(0)} \left\{ (b_1^{(1)})^2 + \frac{1}{3} n_2 a_0^{(0)} a_1^{(0)} \left[b_1^{(1)} + 22(a_1^{(0)})^2 \right] \right\}.
\end{aligned} \tag{14}$$

If $n_2 a_0^{(0)} \lesssim -2a_1^{(0)}$ (i.e., the initial field is less than or of the order of the critical one), then it is not difficult to verify that the term $u^6 a_0^{(6)}$ from the expansion for

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Fig. 1

Figure 1: Fig. 1

a_0 will be considerably smaller than the preceding term ($u^4 a_0^{(4)}$), if the inequality

$$u \ll (ka)^2, \tag{15}$$

is satisfied, where a is the initial radius of the beam ($-2a_1^{(0)} = (ka)^{-2}$). Therefore, under condition (15), the approximate equality

$$a_0 \simeq a_0^{(0)} + u^2 a_0^{(2)} + u^4 a_0^{(4)}. \tag{16}$$

is valid.

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* Let us note that, in general, to determine the coefficient $a_0^{(n)}$ the first $2n-1$ equations of system (10) are necessary, and to determine the coefficient $a_1^{(n)}$, $(2n+1)$ of these equations are necessary.

Let us consider in more detail the case when the initial field is close to the critical one, i.e., when $|\Delta| \ll 1$, where

$$\Delta = E_0^2/E_{\text{cr}}^2 - 1 \tag{17}$$

$$\left(E_0^{(2)} = a_0^{(0)}, \quad E_{\text{cr}}^2 = 1/n_2(ka)^2\right).$$

In this case expressions (14) are simplified and equality (16) takes the form

$$\frac{E^2}{E_0^2} \equiv \frac{a_0}{a_0^{(0)}} \simeq 1 + \frac{\Delta}{(ka)^4}u^2 - \frac{11}{3(ka)^8}u^4. \tag{18}$$

Fig. 1

This expression determines the ratio of the pump field on the beam axis $E^2(u)$ to the initial field on the axis E_0^2. The family of corresponding curves for different values of Δ is shown in Fig. 1. If the initial field is less than the critical one ($\Delta < 0$), then the curves decrease monotonically. If the field E_0 is greater than the critical one, then the axial field E increases up to the maximum value E_m, determined by the equality $E_m^2/E_0^2 = \Gamma(\Delta)$, where

$$\Gamma(\Delta) = 1 + {}^3/_{44}\Delta^2. \tag{19}$$

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In this case the field maximum is reached at the distance l_m, determined by the formula

$$l_m/a = ka\sqrt{3\Delta/22}. \tag{20}$$

For $z > l_m$ the axial field decreases. It is not difficult to verify that in the interval $z < l_m/\sqrt{3}$ the axial field $E(z)$ remains greater than the critical field (12), defined in the same section z; at $z = l_m/\sqrt{3}$ the equality $E^2(z) = E^2_{\text{cr}}(z)$ holds, and for $z > l_m/\sqrt{3}$ the axial field becomes less than the critical one $\left(E^2(z) < E^2_{\text{cr}}(z)\right)$. It is also interesting to note that if the initial field is equal to the critical one, then the axial field decreases with distance from the boundary of the medium. Throughout the entire process of beam propagation, redistribution of its intensity over the transverse cross section with a change in the initial shape plays a substantial role.

Let us now consider the case when the initial field is greater than the critical one $\left(a_0^{(0)} \gg -2a_1^{(0)}\right)$. In this case, from equalities (14) and (13), one obtains the expansion

$$\frac{a_0}{a_0^{(0)}} = 1 + \frac{n_2 a_0^{(0)}}{(ka)^2} u^2 + \frac{\left(n_2 a_0^{(0)}\right)^2}{(ka)^4} u^4 + \dots, \tag{21}$$

from which follows the expression for the characteristic length of the initial change of the axial field:

$$l_x = a/\sqrt{n_2 E_0^2}. \tag{22}$$

This expression was obtained in work [4] from physical considerations. In order to determine the further behavior of the beam, a large number of terms in the expansion of a_0 in a series in powers of u is necessary, which, in turn, requires a large number of equations of the system (8)–(10). In this case it is necessary to resort to a numerical solution of equation (4).

In conclusion, I express my deep gratitude to A. M. Prokhorov for suggesting the topic and for useful discussions.

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Received
23 XI 1966

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Note: Figure translations are in progress. See original paper for figures.

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