SCATTERING OF AN OBLIQUELY INCIDENT PLANE ELECTROMAGNETIC WAVE BY TWO COAXIAL CIRCULAR DISKS

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Abstract

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SCATTERING OF AN OBLIQUELY INCIDENT PLANE ELECTROMAGNETIC WAVE BY TWO COAXIAL CIRCULAR DISKS

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1. The problem of wave scattering by a single circular disk has been the subject of works by many authors (a bibliography of such works may be found in [1–7]); the problem of diffraction of waves by two (or a larger number of) circular disks has been considered by several authors, for example [8–16]. In [8–10] it is solved under the assumption that the disks are excited by a scalar wave, and only in [11–16] is the solution of the electrodynamic problem discussed. Thus, in [16] the problem of diffraction of a plane electromagnetic wave by two coaxial perfectly conducting circular disks is solved under the assumption that the wave propagates along the common axis of symmetry of the disks (the case of normal incidence of the wave). The rigorous solution of this problem in [16] is reduced to the solution of a system of two Fredholm integral equations of the second kind. In [11–15] the solution of the analogous problem is obtained as a particular case of the solution of the more general problem of the excitation of two coaxial disks by the field of a dipole source (electric or magnetic) located on the axis of the disks. The rigorous solution there is found by the method of separation of variables. It is reduced to the solution of infinite systems of linear equations for the coefficients of the expansion of the desired functions in series of proper spheroidal wave functions.

The present paper is devoted to a rigorous solution of the problem of diffraction of a plane electromagnetic wave obliquely incident on the surfaces of the disks. This problem is also solved by us on the basis of the classical method of separation of variables in coordinates of a flattened spheroid. As will be shown below, the problem of excitation of the disks by the field of an electric dipole located at an arbitrary point of space can be solved in a similar way, provided that the dipole moment is parallel to the surfaces of the disks.

2. Let two coaxial disks with radii $\rho_{-1} = a_{-1}$ and $\rho_{+1} = a_{+1}$ (see Fig. 1) be situated in an unbounded homogeneous and isotropic space with electromagnetic constants ε, μ, σ $(\sigma = 0)$, and be excited by the field of a plane electromagnetic wave

$$\vec{E}^{\,0} = \vec{A} e^{ik(\vec{n}\vec{R}) - i\omega t} \left(k = \frac{\omega}{c}\sqrt{\varepsilon\mu}, \quad \frac{\omega}{c} = k_0 \right) \tag{1}$$

with amplitude $A = |\vec{A}| = \text{const}$, propagating in the direction of the unit vector $\vec{n}$, which makes a certain angle α with the positive direction of the axis Oz, coinciding with the axis of symmetry of the disks. The disks are at a distance l from one another. The origin of the coordinate system $Oxyz$ is at the point O, lying midway between the disks on their axis. The plane Oxz is parallel to the vector $\vec{n}$, and wave (1) is polarized so that its electric-intensity vector $\vec{E}^{\,0}$ is 1) parallel to the plane Oxz, 2) perpendicular to the plane Oxz (it is obvious that the solution of the problem for an arbitrary orientation of the vector $\vec{E}^{\,0}$ with respect to the plane Oxz may be obtained as a linear combination of the solutions of the two particular problems corresponding to the above-mentioned cases of polarization of wave (1)). In both cases

$$(\vec{n}\vec{R}) = -x\sin\alpha + z\cos\alpha. \tag{2}$$

The problem is to find the secondary electromagnetic field $\vec{E}^{\,1}, \vec{H}^{\,1}$ scattered by the disks.

To solve the problem by the method of separation of variables, two systems of local coordinates are introduced: $x_s, y_s, z_s,\ s = \pm 1$, associated with each of the disks in such a way that their axes $O_s z_s,\ s = \pm 1$, coincide with Oz, while the remaining axes have the same directions as Ox, Oy, respectively. In the local coordinates of the s-th disk the equation of the incident wave (1) is written in the form $(\vec{R} = \vec{R}_{0s} + \vec{R}_s)$

$$\vec{E}^{\,0} = \vec{A}\psi e^{-i\omega t}, \quad \text{where } \psi = e^{ik(\vec{n}\vec{R})} = e^{ik(\vec{n}\vec{R}_{0s}) + ik(\vec{n}\vec{R}_s)}. \tag{3}$$

Here $\vec{R}_{0s}$ is the radius vector joining the center of the s-th disk O_s to the point O; $\vec{R}_s$ is the radius vector of the observation point. In (3), moreover,

$$(\vec{n}\vec{R}_{0s}) = s\,l_0\cos\alpha, \quad l_0 = l/2,$$

$$s = \pm 1, \tag{4}$$

$$(\vec{n}\vec{R}_s) = -x_s\sin\alpha + z_s\cos\alpha.$$

Fig. 1

It is obvious that in the case when $\vec{E}^{\,0} \perp$ pl. Oxz,

$$\vec{E}^{\,0} = A\{0, \psi, 0\}e^{-i\omega t},$$

$$\vec{H}^{\,0} = \sqrt{\frac{\varepsilon}{\mu}}\,A\{-\psi\cos\alpha,\ 0,\ -\psi\sin\alpha\}e^{-i\omega t}, \tag{5}$$

and in the case when $\vec{E}^{\,0} \parallel$ pl. Oxz,

$$\vec{E}^{\,0} = A\{\psi\cos\alpha,\ 0,\ \psi\sin\alpha\}e^{-i\omega t},$$

$$\vec{H}^{\,0} = \sqrt{\frac{\varepsilon}{\mu}}\,A\{0, \psi, 0\}e^{-i\omega t}. \tag{6}$$

The indicated cases of polarization of wave (1), on the basis of (5), (6), correspond to the electric Hertz vectors of the primary field

$$\vec{\Pi}^{\,0} = \frac{A}{k^2}\{0, \psi, 0\} \tag{7*}$$

[*] The factor $\exp(-i\omega t)$ is omitted here and below.

and

$$\vec{\Pi}^{\,0} = \frac{A}{k^2\cos\alpha}\{\psi,\ 0,\ 0\} \tag{8}$$

respectively.

To determine the secondary electromagnetic field, the Hertz vector $\vec{\Pi}^{\,1} = \{\Pi_x^1,\, \Pi_y^1,\, 0\}$, corresponding to this field and parallel to the planes of the disks, is introduced; moreover, it is assumed that $\Pi_x^1 \equiv 0$ $(\Pi_y^1 \neq 0)$ in the case (7), and $\Pi_y^1 = 0$ $(\Pi_x^1 \neq 0)$ in the case (8).

The total electromagnetic field $\vec{E}, \vec{H}$ is found through the Hertz vector $\vec{\Pi} = \vec{\Pi}^{\,0} + \vec{\Pi}^{\,1}$ from the relations

$$\left.\begin{aligned} \vec{E} &= \operatorname{grad}\operatorname{div}\vec{\Pi} + k^2\vec{\Pi}, \\ \vec{H} &= -ik_0\varepsilon\ \operatorname{rot}\vec{\Pi}. \end{aligned}\right\} \tag{9}$$

In the mathematical formulation, the problem consists in finding the potentials $u = \Pi_x^1$, $v = \Pi_y^1$ of the scattered field, satisfying in Cartesian rectangular coordinates the homogeneous scalar Helmholtz wave equation

$$\Delta U + k^2 U = 0, \tag{10}$$

where U denotes either u or v, the prescribed boundary conditions on the surface of each disk and on their edges, and the radiation condition at infinity. The boundary conditions for the functions u, v are obtained from the requirement that the tangential components of the vector $\vec{E}$ of the total field vanish on the surfaces of the disks. In the general case (the vector $\vec{\Pi}$ is parallel to the surfaces of the disks), these conditions are written in the form

$$\left.\begin{aligned} f = \frac{1}{k^2}\frac{\partial\Phi}{\partial x_s}, \qquad g = \frac{1}{k^2}\frac{\partial\Phi}{\partial y_s}, \\ \Delta\Phi + k^2\Phi = 0, \qquad\qquad\qquad\qquad \\ z_s = 0, \quad x_s^2 + y_s^2 < a_s^2, \quad s = \pm 1, \end{aligned}\right\} \tag{11}$$

where

$$f = u^0 + u, \qquad g = v^0 + v, \qquad \Phi = -\frac{\partial f}{\partial x_s} - \frac{\partial g}{\partial y_s},$$

$$u^0 = \Pi_x^0, \qquad v^0 = \Pi_y^0.$$

If one passes from the local coordinates x_s, y_s, z_s to the local coordinates of an oblate spheroid ξ_s, η_s, φ_s ($\varphi_s = \varphi$, $s = \pm 1$), which contain the corresponding disks as degenerate cases, then in these coordinates the surface of the s-th disk will be determined by the equation $\xi_s = 0$, and the condition on the edge ($x_s^2 + y_s^2 = a_s^2$, $z_s = 0$) of each disk can be written in the form [17]

$$\left.\begin{array}{c} \dfrac{\partial}{\partial\xi_s} \\ \dfrac{\partial}{\partial\eta_s} \end{array}\right\} (u\cos\varphi + v\sin\varphi) = 0, \qquad \xi_s = \eta_s = 0, \quad s = \pm 1. \tag{12}$$

3. As in the analogous problem referred to the case of one disk [17] (see also [6]), we split the field scattered by the disks into two parts, the first of which is determined by the vector $\vec{\Pi}_1 = \{\Pi_{1,x}, \Pi_{1,y}, 0\}$, and the second part by the vector $\vec{\Pi}_2 = \{\Pi_{2,x}, \Pi_{2,y}, 0\}$, so that

$$\vec{\Pi}^1 = \vec{\Pi}_1 + \vec{\Pi}_2. \tag{13}$$

The vector $\vec{\Pi}_1$ is required, on the surfaces of the disks, together with the vector $\vec{\Pi}^0$ of the primary field, to satisfy the condition

$$\left(\vec{\Pi}^0 + \vec{\Pi}_1\right)_{\xi_s=0} = 0, \quad s = \pm 1. \tag{14}$$

Then condition (11) for $\xi_s = 0$, $s = \pm 1$, will be satisfied only by the second part of the scattered field, $\vec{\Pi}_2$.

Expanding the function ψ in a series in wave spheroidal functions [6, 18] in the local coordinates of the s-th disk, in the case when $\vec{E}^0 \perp$ plane Oxz, we obtain

$$v^0 = \frac{2A}{k^2} e^{iskl_0 \cos\alpha} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \frac{(-1)^m S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)} \times$$

$$\times R^{(1)}_{|m|n}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{15}$$

$$(u^0 = 0),$$

and in the case when $\vec{E}^0 \parallel$ plane Oxz,

$$u^0 = \frac{2Ae^{iskl_0 \cos\alpha}}{k^2 \cos\alpha} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \frac{(-1)^m S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)} \times$$

$$\times R^{(1)}_{|m|n}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{16}$$

$$(v^0 = 0).$$

The spheroidal wave functions $S_{mn}(-ic, \eta)$, $R_{mn}(-ic, \xi)$ and the quantity $N_{mn}(-ic)$, determining the norm of the angular functions $S_{mn}(-ic)$, are the same as in [6]; $c_s = ka_s = 2\pi a_s/\lambda,\ s = \pm 1$.

Let us first find the first part of the scattered field—the vector $\vec{\Pi}_1$. Depending on the type of polarization of the incident wave, the vector $\vec{\Pi}_1$ is sought in the form

$$v_1 = \Pi_{1,y} =$$

$$= \frac{2A}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a^s_{mn} R^{(3)}_{|m|n}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{17}$$

$$(\Pi_{1,x} = 0)$$

or in the form

$$u_1 = \Pi_{1,x} =$$

$$= \frac{2A}{k^2 \cos\alpha} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b^s_{mn} R^{(3)}_{|m|n}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{18}$$

($\Pi_{1,y} = 0$), respectively. Applying the addition theorem for spheroidal wave functions, written in the form

$$R^{(3)}_{|m|n}(-ic_{-s},\, i\xi_{-s})S_{mn}(-ic_{-s},\, \eta_{-s}) =$$

$$= \sum_{q=|m|}^{\infty} Q_{mqmn}(c_s,\, c_{-s};\, l,\, \theta_{-s,s})R^{(1)}_{|m|q}(-ic_s,\, i\xi_s)S_{mq}(-ic_s,\, \eta_s), \tag{19}$$

where

$$Q_{mqmn} = \frac{2i^{\,q-n}}{N_{mq}(-ic_s)} \sum_{\lambda=0,1}^{\infty}{}' \sum_{\tau=0,1}^{\infty}{}' d^{mq}_{\lambda}(-ic_s)d^{mn}_{\tau}(-ic_{-s})\times$$

$$\times \sum_{\sigma=|\lambda-\tau|}^{\lambda+\tau+2m} i^{\sigma} b_{\sigma}^{(\lambda+m,\, m,\, \tau+m,\, m)} h^{(1)}_{\sigma}(kl)P_{\sigma}(\cos\theta_{-s,s}), \tag{20}$$

$$\theta_{-1,1} = 0, \qquad \theta_{+1,-1} = \pi$$

($b^{(nmpq)}_{\sigma}$ are the expansion coefficients

$$P^m_n(\cos\theta)P^p_q(\cos\theta) = \sum_{\sigma=|n-q|}^{n+q} b^{(nmpq)}_{\sigma} P^{m-p}_{\sigma}(\cos\theta) \tag{21}$$

of the product of associated Legendre functions in terms of the same functions; see, for example, [19]), and $h^{(1)}_{\sigma}(x)$ is the spherical Bessel function,

$$h^{(1)}_{\sigma}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{\sigma+1/2}(x),$$

and using the orthogonality property of the angular spheroidal functions on the interval $[-1, +1]$, from the boundary condition (14) we obtain, for the coefficients a^s_{mn} and b^s_{mn} of the series (17) and (18), infinite systems of linear equations in normal form, whose right-hand sides and matrix elements (except for the diagonal ones, equal to unity) will contain as a factor the function $R^{(1)}_{|m|n}(-ic_s,\, i0)$, which is equal to zero for odd values of $n \pm m$. Therefore it is immediately found from the systems that

$$a^s_{mn} = 0 \quad \text{and} \quad b^s_{mn} = 0 \quad \text{for odd } n \pm m; \quad s = \pm 1. \tag{22}$$

If, in the systems, the unknowns a^s_{mn} and b^s_{mn} are replaced by the new unknowns A^s_{mn} and B^s_{mn} according to the formulas

$$\begin{aligned} a^s_{mn} &= R^{(1)}_{|m|n}(-ic_s,\, i0)\, A^s_{mn}, \\ b^s_{mn} &= R^{(1)}_{|m|n}(-ic_s,\, i0)\, B^s_{mn}, \end{aligned} \qquad \{ \ (n \pm m)\text{—even}, \tag{23}$$

then for the new unknowns one obtains the infinite systems

$$A^s_{mn} + \sum_{q=|m|}^{\infty} \alpha^{(-s,s)}_{mnq} A^{-s}_{mq} = f^s_{mn} \tag{24}$$

and

$$B^s_{mn} + \sum_{q=|m|}^{\infty} \alpha^{(-s,s)}_{mnq} B^{-s}_{mq} = f^s_{mn} \tag{25}$$

$$(|m| \le n;\ \ n = 0, 1, \ldots;\ \ s = \pm 1)$$

with matrix elements

$$\alpha^{(-s,s)}_{mnq} = \frac{R^{(1)}_{|m|q}(-ic_s, i0)}{R^{(3)}_{|m|n}(-ic_s, i0)}\, Q_{mnq}(c_s, c_{-s}; l, \theta_{-s,s}) \tag{26}$$

and with free terms

$$f^s_{mn} = \frac{(-1)^m i^n e^{iskl_0 \cos\alpha} S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)\, R^{(3)\prime}_{|m|n}(-ic_s, i0)}. \tag{27}$$

The systems (24), (25) are identical ($A^s_{mn} \equiv B^s_{mn}$), and therefore in what follows we shall consider only one of them—the system (24). Its infinite determinant Δ_m can be written, for each m, $|m| \le n$, in the form

$$\Delta_m = \left| \begin{array}{ccccccccc} 1 & \alpha^-_{m,m} & 0 & \alpha^-_{m,m+1} & 0 & \alpha^-_{m,m+2} & 0 & \alpha^-_{m,m+3} & 0 \\ \alpha^+_{m,m} & 1 & \alpha^+_{m,m+1} & 0 & \alpha^+_{m,m+2} & 0 & \alpha^+_{m,m+3} & 0 & \alpha^+_{m,m+4} \\ 0 & \alpha^-_{m+1,m} & 1 & \alpha^-_{m+1,m+1} & 0 & \alpha^-_{m+1,m+2} & 0 & \alpha^-_{m+1,m+3} & 0 \\ \alpha^+_{m+1,m} & 0 & \alpha^+_{m+1,m+1} & 1 & \alpha^+_{m+1,m+2} & 0 & \alpha^+_{m+1,m+3} & 0 & \alpha^+_{m+1,m+4} \\ 0 & \alpha^-_{m+2,m} & 0 & \alpha^-_{m+2,m+1} & 1 & \alpha^-_{m+2,m+2} & 0 & \alpha^-_{m+2,m+3} & 0 \\ \alpha^+_{m+2,m} & 0 & \alpha^+_{m+2,m+1} & 0 & \alpha^+_{m+2,m+2} & 1 & \alpha^+_{m+2,m+3} & 0 & \alpha^+_{m+2,m+4} \\ 0 & \alpha^-_{m+3,m} & 0 & \alpha^-_{m+3,m+1} & 0 & \alpha^-_{m+3,m+2} & 1 & \alpha^-_{m+3,m+3} & 0 \\ \alpha^+_{m+3,m} & 0 & \alpha^+_{m+3,m+1} & 0 & \alpha^+_{m+3,m+2} & 0 & \alpha^+_{m+3,m+3} & 1 & \alpha^+_{m+3,m+4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right. \tag{28}$$

where

$$\alpha^-_{n,q} = \alpha^{(-1,+1)}_{mnq}, \qquad \alpha^+_{n,q} = \alpha^{(+1,-1)}_{mnq},$$

and the system itself can be written in normal form

$$x^m_i + \sum_{j=1}^{\infty} c^m_{ij} x^m_j = f^m_i \qquad (i = 1, 2, ...), \tag{29}$$

if we put

$$x^m_{2i-1} = A^{+1}_{m,m+i-1}, \qquad f^m_{2i-1} = f^{+1}_{m,m+i-1},$$

$$x^m_{2i} = A^{-1}_{m,m+i-1}, \qquad f^m_{2i} = f^{-1}_{m,m+i-1}, \tag{30}$$

$$c^m_{2j-1,\,2i} = \alpha^{(-1,+1)}_{m,m+j-1,m+i-1}, \qquad c^m_{2j,\,2i-1} = \alpha^{(+1,-1)}_{m,m+j-1,m+i-1},$$

$$c^m_{2j-1,\,2i-1} = c^m_{2j,\,2i} = 0 \qquad (i, j = 1, 2, ...). \tag{31}$$

In the case of normal incidence of the plane wave (1) on the disks, when $\alpha = 0$,

$$f^s_{mn} = 0, \quad \text{if } m \neq 0,$$

and then

$$A^s_{mn} = B^s_{mn} = 0, \quad \text{if } m \neq 0.$$

In this case

$$u_1 = v_1 = \frac{A}{k^2} \sum_{s=\pm 1} \sum_{n=0,2,\dots} A^s_{0n} R^{(1)}_{0n}(-ic_s, i0) \times$$

$$\times R^{(3)}_{0n}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_{1s}), \tag{32}$$

and for the primary field

$$u^0 = v^0 = \frac{2Ae^{iskl_0}}{k^2} \sum_{n=0}^{\infty} \frac{i^n S_{0n}(-ic_s, 1)}{N_{0n}(-ic_s)} \times$$

$$\times R^{(1)}_{0n}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_{1s}). \tag{33}$$

The coefficients A^s_{0n}, $s = \pm 1$, are found from the system (24) ((29)) with $m = 0$, which for $m = 0$, $\alpha = 0$, will be similar to the infinite system of linear equations considered in [12, 14], where its unique solvability by the method of truncation is proved. In the general case, the coefficients $A^{\pm 1}_{mn}$ are also found from the system (24) ((29)) by the method of truncation. Moreover, if N is the order of truncation of the system $(n, q = |m|, |m| + 1, \dots, N)$, then, in order to find the approximate values $A^{\pm 1}_{mn}$, $n = |m|, |m| + 1, \dots, N$, for each m one must solve a system of $2(N - |m| + 1)$ equations of the form

$$A^s_{mn} + \sum_{q=|m|}^{N} a^{(-s,s)}_{mnq} A^{-s}_{mq} = f^s_{mn}, \quad s = \pm 1, \quad n = |m|, |m| + 1, \dots, N \tag{34}$$

or of the form

$$x^m_i + \sum_{j=1}^{2(N-|m|+1)} c^m_{ij} x^m_j = f^m_i, \quad i = 1, \dots, 2(N - |m| + 1), \tag{35}$$

if, instead of the system (24), one considers the system (29).

The accuracy of the calculation can be determined by comparing the approximate values A_{mn} obtained from (34) ((35)) for various increasing values of N, the order of truncation. As the initial value of N it is advisable to take a number approximately equal to c_0, where $c_0 = \max_{s=\pm 1}(c_s)$ (this statement is based on the consideration that the series (17), (18), after replacing their coefficients by new ones according to formulas (23), have the same convergence as the corresponding diffraction series in the analogous problem for a single disk, in the computation of which one usually restricts oneself to the first n terms, where $n \approx c_0 = ka$).

By determining A^s_{mn}, the problem of finding the first part of the scattered field is solved.

The second part of the scattered field, determined by the vector $\vec{\Pi}_2$, is sought, depending on the type of polarization of the wave (1), in the form

$$v_2 = \Pi_{2,y} =$$

$$= \frac{2A}{k^2} \sum_{s=\pm1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{a}^s_{mn} R^{(3)}_{|m|n}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{36}$$

($\bar{\Pi}_{2,x} = 0$), when $\vec{E}^{\,0} \perp$ the plane Oxz, or in the form

$$u_2 = \Pi_{2,x} =$$

$$= \frac{2A}{k^2 \cos\alpha} \sum_{s=\pm1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{b}^s_{mn} R^{(3)}_{|m|n}(-ic_s,\ i\xi_s) S_{mn}(-ic_s,\ \eta_s) e^{im\varphi} \tag{37}$$

($\Pi_{2,y} = 0$), when $\vec{E}^{\,0} \parallel$ the plane Oxz. The coefficients of the series are found as follows.

As in the problem for one disk [18], let us first expand the function Φ from (11) on the surface of the s-th disk in a series in cylindrical wave functions

$$\Phi = \sum_{m=-\infty}^{\infty} c^s_m J_m(k\rho_s) e^{im\varphi}, \tag{38}$$

(in local coordinates $\rho_s,\ \varphi_s = \varphi$ with origin at the center of the disk O_s) and then substitute (38) into the boundary conditions (11). As a result, after applying the formula [20]

$$J'_m(x) \pm \frac{m}{x} J_m(x) = \pm J_{m\mp1}(x)$$

we obtain the relations

$$v_2 = \frac{ik}{2} \sum_{m=-\infty}^{\infty} x^s_m J_m(k\rho_s) e^{im\varphi} \tag{39}$$

or

$$u_2 = \frac{k}{2} \sum_{m=-\infty}^{\infty} y^s_m J_m(k\rho_s) e^{im\varphi}, \tag{40}$$

where

$$x^s_m = c^s_{m+1} + c^s_{m-1}, \qquad y^s_m = c^s_{m+1} - c^s_{m-1}. \tag{41}$$

Replacing in (39) and (40) their left-hand sides by their expansions (36) and (37) and applying the addition theorem (19), we find, after using the formula [6]

$$J_m(k\rho) = 2\sum_{n=|m|}^{\infty} i^{\,n-m}\frac{S_{mn}(-ic,0)}{N_{mn}(-ic)}S_{mn}(-ic,\eta)\,R^{(1)}_{|m|n}(-ic,i\xi),$$

that the coefficients $\bar{a}^s_{mn}$ and $\bar{b}^s_{mn}$ must be solutions of infinite systems of linear equations, from which it can immediately be established that

$$\bar{a}^s_{mn} = \bar{b}^s_{mn} = 0, \quad \text{if } (n \pm m) \text{ is odd.} \tag{42}$$

Carrying out in the systems the replacement of the unknowns $\bar{a}^s_{mn}$, $\bar{b}^s_{mn}$ by the new $\bar{A}^s_{mn}$, $\bar{B}^s_{mn}$ according to the formulas

$$\left.\begin{aligned} \bar{a}^s_{mn} &= R^{(1)}_{|m|n}(-ic_s,\ i0)\,\bar{A}^s_{mn},\\ \bar{b}^s_{mn} &= R^{(1)}_{|m|n}(-ic_s,\ i0)\,\bar{B}^s_{mn}, \end{aligned}\right\} \qquad (n \pm m)\text{—even}, \tag{43}$$

we obtain from them, for $\overline{A}^s_{mn}$ and $\overline{B}^s_{mn}$, infinite systems of equations of the form

$$\overline{A}^s_{mn} + \sum_{q=|m|}^{\infty} \alpha^{(-s,s)}_{mnq}\overline{A}^{-s}_{mq} = F^s_{mn}, \tag{44}$$

$$\overline{B}^s_{mn} + \sum_{q=|m|}^{\infty} \alpha^{(-s,s)}_{mnq}\overline{B}^{-s}_{mq} = \Phi^s_{mn} \tag{45}$$

$$n = |m|,\ |m|+1,\ \ldots; \quad s = \pm 1$$

with the same matrix elements as in the system (24). The determinants of the systems (44), (45) can be written in the form (28), and the systems themselves, whose right-hand sides are given by the expressions

$$\begin{aligned} F^s_{mn} &= -ix^s_m p^s_{mn},\\ \Phi^s_{mn} &= -y^s_m p^s_{mn}\cos\alpha, \end{aligned} \tag{46}$$

where

$$p^s_{mn} = \frac{k^3}{2A}\, i^{\,n-m}\, \frac{S_{mn}(-ic_s, 0)}{N_{mn}(-ic_s)\, R^{(3)}_{|m|n}(-ic_s, i0)},$$

can be written in the normal form (29). The right-hand sides (46) contain the unknown quantities x^s_m and y^s_m, $s = \pm 1$. To determine them, the conditions on the disk edges (12), written for the total scattered field, are used. On the basis of the addition theorem (19), from (12) one obtains the relations

$$\sum_{n=|m|}^{\infty} \left(A^s_{mn} + \overline{A}^s_{mn}\right) S_{mn}(-ic_s, 0) = 0, \quad s = \pm 1, \tag{47}$$

when $\vec{E}^{\,0} \perp$ the plane Oxz, and

$$\sum_{n=|m|}^{\infty} \left(B^s_{mn} + \overline{B}^s_{mn}\right) S_{mn}(-ic_s, 0) = 0, \quad s = \pm 1, \tag{48}$$

when $\vec{E}^{\,0} \parallel$ the plane Oxz.

The solutions of the systems (44), (45) can be written in the form

$$\begin{aligned} \overline{A}^s_{mn} &= \alpha^s_{mn} x^s_m + \beta^s_{mn} x^{-s}_m, \quad s = \pm 1, \\ \overline{B}^s_{mn} &= \gamma^s_{mn} y^s_m + \delta^s_{mn} y^{-s}_m, \quad s = \pm 1, \end{aligned} \tag{49}$$

where α^s_{mn}, β^s_{mn}, γ^s_{mn}, and δ^s_{mn} are certain expressions independent of x^s_m and y^s_m, whose values for each m are found from (44) and (45). Therefore, after substitution of (49) into the corresponding relations (47), (48), for determining the quantities x^s_m and y^s_m two systems of equations are obtained:

$$\alpha^s_m x^s_m + \beta^s_m x^{-s}_m = \tau^s_m, \quad s = \pm 1, \tag{50}$$

and

$$\gamma^s_m y^s_m + \delta^s_m y^{-s}_m = \sigma^s_m, \quad s = \pm 1, \tag{51}$$

where

$$\tau^s_m = -\sum_{n=|m|}^{\infty} A^s_{mn} S_{mn}(-ic_s, 0),$$

$$\alpha_m^s = \sum_{n=|m|}^{\infty} \alpha_{mn}^s S_{mn}(-ic_s, 0),$$

$$\beta_m^s = \sum_{n=|m|}^{\infty} \beta_{mn}^s S_{mn}(-ic_s, 0),$$

$$\sigma_m^s = \tau_m^s, \quad \gamma_m^s = -i\cos\alpha \cdot \alpha_m^s, \quad \delta_m^s = -i\beta_m^s \cos\alpha$$

$$(\gamma_{mn}^s = -i\alpha_{mn}^s \cos\alpha, \quad \delta_{mn}^s = -i\beta_{mn}^s \cos\alpha).$$

Solving the systems (50), (51), we find that

$$x_m^s = \frac{\tau_m^s \alpha_m^{-s} - \tau_m^{-s}\beta_m^s}{\alpha_m^s \alpha_m^{-s} - \beta_m^s \beta_m^{-s}}, \quad s = \pm 1, \tag{52}$$

$$y_m^s = \frac{i}{\cos\alpha} x_m^s, \quad s = \pm 1, \tag{53}$$

and thus the values of the unknowns $\bar{A}_{mn}^s$ and $\bar{B}_{mn}^s$ are completely determined, and through them also the second part of the scattered field, determined by the vector $\vec{\Pi}_2$.

In the case of normal incidence of wave (1) on the disks, when $\alpha = 0$,

$$x_m^s = y_m^s = 0, \quad \text{if } m \neq 0, \quad s = \pm 1, \quad \text{and } x_0^s = -iy_0^s,$$

and then $\bar{A}_{0n}^s = \bar{B}_{0n}^s$, while

$$u_2 = v_2 = \frac{2A}{k^2} \sum_{s=\pm 1} \sum_{n=0,2,\ldots} \bar{A}_{0n}^s R_{0n}^{(1)}(-ic_s, i0) \times$$

$$\times R_{0n}^{(3)}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_s). \tag{54}$$

If $\alpha \neq 0$, then for the total scattered field

$$u = \Pi_x^1 = u_1 + u_2 = \frac{2A}{k^2 \cos\alpha} \sum_{s=\pm 1} \sum_{n=0,2,\ldots} \sum_{m=-n}^{n} (B_{mn}^s + \bar{B}_{mn}^s) \times$$

$$\times R_{|m|n}^{(1)}(-ic_s, i0) R_{|m|n}^{(3)}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi}, \tag{55}$$

when $\vec{E}^0 \parallel$ plane Oxz, and

$$v = \Pi_y^1 = v_1 + v_2 = \frac{2A}{k^2}\sum_{s=\pm1}\sum_{n=0,2,\dots}\sum_{m=-n}^{n}(A_{mn}^s + \bar{A}_{mn}^s)\times$$

$$\times R_{|m|n}^{(1)}(-ic_s, i0)R_{|m|n}^{(3)}(-ic_s, i\xi_s)S_{mn}(-ic_s, \eta_s)e^{im\varphi}, \tag{56}$$

when $\vec{E}^{\,0} \perp$ plane Oxz. For $\alpha = 0$, $u \equiv v$.

4. The components of the vectors $\vec{E}^{\,1}$, $\vec{H}^{\,1}$ of the electromagnetic field scattered by the disks are found at any point from relations (9), written in one-of the coordinate systems. In particular, in the coordinates of the oblate spheroid ξ, η, φ, from (9) we have:

$$\begin{aligned}
E_\eta^1 &= \frac{1}{a}\sqrt{\frac{(1-\eta^2)(\xi^2+1)}{\xi^2+\eta^2}}\cos\varphi\times\\
&\quad\times\frac{\partial}{\partial\eta}\left\{\frac{\sqrt{1-\eta^2}}{\xi^2+\eta^2}\left(\xi\frac{\partial\Pi_x^1}{\partial\xi} - \eta\frac{\partial\Pi_x^1}{\partial\eta}\right)\right\} - k^2\eta\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\Pi_x^1\cos\varphi,\\
E_\varphi^1 &= \frac{-\sin\varphi}{a^2(\xi^2+\eta^2)}\left(\xi\frac{\partial\Pi_x^1}{\partial\xi} - \eta\frac{\partial\Pi_x^1}{\partial\eta}\right) - k^2\sin\varphi\,\Pi_x^1,\\
H_\eta^1 &= \frac{ik_0\varepsilon\sin\varphi}{a(\xi^2+\eta^2)^{1/2}}\left\{\frac{\xi}{\sqrt{\xi^2+1}}\Pi_x^1 - \frac{\partial}{\partial\xi}\left(\sqrt{1+\xi^2}\,\Pi_x^1\right)\right\},\\
H_\varphi^1 &= \frac{ik_0\varepsilon\sqrt{(1-\eta^2)(\xi^2+1)}}{a(\xi^2+\eta^2)}\cos\varphi\times\\
&\quad\times\left\{\frac{\eta}{\sqrt{1-\eta^2}}\frac{\partial}{\partial\xi}\left(\sqrt{1+\xi^2}\,\Pi_x^1\right) - \frac{\xi}{\sqrt{1+\xi^2}}\frac{\partial}{\partial\eta}\left(\sqrt{1-\eta^2}\,\Pi_x^1\right)\right\},
\end{aligned} \tag{57}$$

if $\vec{E}^{\,0} \parallel$ plane Oxz, and

$$
\begin{aligned}
E^1_\eta &= \frac{1}{a}\sqrt{\frac{(1-\eta^2)(\xi^2+1)}{\xi^2+\eta^2}}\sin\varphi\times \\
&\quad\times\frac{\partial}{\partial\eta}\left\{\frac{\sqrt{1-\eta^2}}{\xi^2+\eta^2}\left(\xi\frac{\partial\Pi^1_y}{\partial\xi}-\eta\frac{\partial\Pi^1_y}{\partial\eta}\right)\right\}-k^2\eta\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\Pi^1_y\sin\varphi, \\
E^1_\varphi &= \frac{-\cos\varphi}{a^2(\xi^2+\eta^2)}\left(\xi\frac{\partial\Pi^1_y}{\partial\xi}-\eta\frac{\partial\Pi^1_y}{\partial\eta}\right)+k^2\Pi^1_y\cos\varphi, \\
H^1_\eta &= \frac{-ik_0\varepsilon\cos\varphi}{a(\xi^2+\eta^2)^{1/2}}\left\{\frac{\xi}{\sqrt{\xi^2+1}}\Pi^1_y-\frac{\partial}{\partial\xi}\left(\sqrt{1+\xi^2}\,\Pi^1_y\right)\right\}, \\
H^1_\varphi &= \frac{ik_0\varepsilon\sqrt{(1-\eta^2)(\xi^2+1)}}{a(\xi^2+\eta^2)}\sin\varphi\times \\
&\quad\times\left\{\frac{\eta}{\sqrt{1-\eta^2}}\frac{\partial}{\partial\xi}\left(\sqrt{1+\xi^2}\,\Pi^1_y\right)-\frac{\xi}{\sqrt{1+\xi^2}}\frac{\partial}{\partial\eta}\left(\sqrt{1-\eta^2}\,\Pi^1_y\right)\right\},
\end{aligned}
\tag{58}
$$

if $\vec{E}^{\,0}\perp$ plane Oxz.

In the case when $\vec{E}^{\,0}\parallel$ plane Oxz, in the wave-zone approximation in spherical coordinates r,θ,φ with origin at the point O, we find that

$$
\begin{aligned}
E^1_\eta &= \sqrt{\frac{\mu}{\varepsilon}}\,H^1_\varphi = -A\,\frac{e^{ikr}}{ikr}\,S_1(\theta,\varphi;\alpha), \\
E^1_\varphi &= -\sqrt{\frac{\mu}{\varepsilon}}\,H^1_\eta = -A\,\frac{e^{ikr}}{ikr}\,S_2(\theta,\varphi;\alpha),
\end{aligned}
\tag{59}
$$

where the amplitude functions are equal to

$$
S_1(\theta,\varphi;\alpha) =
$$

$$
=\frac{2\cos\theta\cos\varphi}{\cos\alpha}\sum_{s=\pm1}e^{-iskl_0\cos\theta}\sum_{n=0,2,\ldots}\sum_{m=-n}^{n}i^n\left(B^s_{mn}+\bar{B}^s_{mn}\right)\times
$$

$$
\times R^{(1)}_{|m|n}(-ic_s,i0)\,S_{mn}(-ic_s,\eta_s)e^{im\varphi}, \tag{60}
$$

$$
S_2(\theta,\varphi;\alpha)=\frac{\operatorname{tg}\varphi}{\cos\theta}\,S_1(\theta,\varphi;\alpha). \tag{61}
$$

The value σ_{b} of the backscattering cross section (in the direction toward the source: $\theta=\pi-\alpha,\ \varphi=0$) for the diffracting system consisting of two disks can be calculated by the formula

$$\sigma_{\mathrm{b}} = \frac{4\pi}{k^2}\,|S_1(\pi-\alpha,0;\alpha)|^2, \tag{62}$$

where

$$S_1(\pi-\alpha,0;\alpha) = -2\sum_{s=\pm1} e^{iskl_0\cos\alpha} \sum_{n=0,2,\ldots}\sum_{m=-n}^{n} i^n\left(B^s_{mn}+\bar{B}^s_{mn}\right)\times$$

$$\times R^{(1)}_{|m|n}(-ic_s,i0)\,S_{mn}(-ic_s,\cos\alpha).$$

For $\alpha = 0$

$$S_1(\pi,0;0) = -2\sum_{s=\pm1} e^{iskl_0} \sum_{n=0,2,\ldots} i^n\left(A^s_{0n}+\bar{A}^s_{0n}\right)\times$$

$$\times R^{(1)}_{0n}(-ic_s,i0)\,S_{0n}(-ic_s,1).$$

With the aid of the addition theorem and relations (24), (25)–(27), (44)–(46), for the potential $f = \Pi_x$ of the total electromagnetic field in the vicinity of the s-th disk one obtains the expression

$$f = \frac{2A}{k^2\cos\alpha}\sum_{n=0,2,\ldots}\sum_{m=-n}^{n}\left(B^s_{mn}+\bar{B}^s_{mn}\right)\times$$

$$\times\left\{R^{(1)}_{|m|n}(-ic_s,i0)R^{(3)}_{|m|n}(-ic_s,i\xi_s)-\right.$$

$$\left.-R^{(1)}_{|m|n}(-ic_s,i\xi_s)R^{(3)}_{|m|n}(-ic_s,i0)\right\}S_{mn}(-ic_s,\eta_s)e^{im\varphi}- \tag{63}$$

$$-k\sum_{n=0,2,\ldots}\sum_{m=-n}^{n} i^{n-m}y^s_m\,\frac{S_{mn}(-ic_s,0)}{N_{mn}(-ic_s)}\times$$

$$\times R^{(1)}_{|m|n}(-ic_s,i\xi_s)S_{mn}(-ic_s,\eta_s)e^{im\varphi}$$

($\xi_{-s,s} > \xi_s$, $s = \pm1$, where $\xi_{-s,s}$ is the radial coordinate of the point O_s in the coordinates of the $-s$-th disk).

On the basis of (63) and (58) we find the law of distribution of the density of the surface currents induced on the s-th disk:

$$j_{\eta_s} = \frac{iAc\sqrt{\dfrac{\varepsilon}{\mu}}\cos\varphi}{2\pi\eta_s c_s^2\cos\alpha}\,\Psi_1(\eta_s,\varphi),$$

$$j_{\varphi_s} = -\frac{iAc\sqrt{\dfrac{\varepsilon}{\mu}}\sin\varphi}{2\pi\eta_s c_s^2\cos\alpha}\,\Psi_1(\eta_s,\varphi), \tag{64}$$

where

$$\Psi_1^r(\eta_s,\ \varphi) = \sum_{n=0,2,\ldots}\sum_{m=-n}^{n}\left(B^s_{mn} + \overline{B}^s_{mn}\right)S_{mn}(-ic_s,\eta_s)e^{im\varphi}.$$

In the case when $\vec{E}^{\,0} \perp$ pl. Oxz, in the wave-zone approximation in spherical coordinates with origin at the point O,

$$E^1_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\,H^1_\eta = A\,\frac{e^{ikr}}{ikr}\,S_3(\theta,\varphi;\alpha),$$

$$E^1_\eta = \sqrt{\frac{\mu}{\varepsilon}}\,H^1_\varphi = -A\,\frac{e^{ikr}}{ikr}\,S_4(\theta,\varphi;\alpha), \tag{65}$$

where the amplitude functions are

$$S_4(\theta,\varphi;\alpha) = 2\cos\theta\cos\varphi\sum_{s=\pm1}e^{-iskl_0\cos\theta}\sum_{n=0,2,\ldots}\sum_{m=-n}^{n} i^n\left(A^s_{mn} + \overline{A}^s_{mn}\right)\times$$

$$\times R^{(1)}_{|m|n}(-ic_s,i0)\,S_{mn}(-ic_s,\eta_s)e^{im\varphi}, \tag{66}$$

$$S_3(\theta,\varphi;\alpha) = -\frac{\operatorname{ctg}\varphi}{\cos\theta}\,S_4(\theta,\varphi;\alpha). \tag{67}$$

The magnitude of the backscattering cross section is found from the formula

$$\sigma_{\mathrm{b}} = \frac{4\pi}{k^2}\,\left|S_3(\pi-\alpha,0;\alpha)\right|^2, \tag{68}$$

where

$$S_3(\pi-\alpha,0;\alpha) = 2\sum_{s=\pm1}e^{iskl_0\cos\alpha}\sum_{n=0,2,\ldots}\sum_{m=-n}^{n} i^n\left(A^s_{mn} + \overline{A}^s_{mn}\right)\times$$

$$\times R^{(1)}_{|m|n}(-ic_s, i0)\, S_{mn}(-ic_s, \cos\alpha).$$

For $\alpha = 0$

$$S_3(\pi, 0; 0) = 2 \sum_{s=\pm 1} e^{iskl_0} \sum_{n=0,2,\ldots} i^n \left(A^s_{0n} + \overline{A}^s_{0n} \right) \times$$

$$\times R^{(1)}_{0n}(-ic_s, i0)\, S_{0n}(-ic_s, 1).$$

For the total field near the s-th disk one obtains the expression

$$g = \Pi_y = \frac{2A}{k^2} \sum_{n=0,2,\ldots} \sum_{m=-n}^{n} \left(A^s_{mn} + \overline{A}^s_{mn} \right) \times$$

$$\times \left\{ R^{(1)}_{|m|n}(-ic_s, i0)\, R^{(3)}_{|m|n}(-ic_s, i\xi_s) - R^{(1)}_{|m|n}(-ic_s, i\xi_s)\, R^{(3)}_{|m|n}(-ic_s, i0) \right\} S_{mn}(-ic_s, \eta_s) e^{im\varphi} - \tag{69}$$

$$-ik \sum_{n=0,2,\ldots} \sum_{m=-n}^{n} i^{n-m} x^s_m \frac{S_{mn}(-ic_s, 0)}{N_{mn}(-ic_s)} \times$$

$$\times R^{(1)}_{|m|n}(-ic_s, i\xi_s)\, S_{mn}(-ic_s, \eta_s) e^{im\varphi},$$

on the basis of which and (58) we find the distribution law for the density of the surface currents on the s-th disk:

$$j_{\eta_s} = \frac{iAc\sqrt{\dfrac{\varepsilon}{\mu}}\sin\varphi}{2\pi\eta_s c_s^2}\, \Psi_2(\eta_s, \varphi),$$

$$j_{\varphi_s} = -\frac{iAc\sqrt{\dfrac{\varepsilon}{\mu}}\cos\varphi}{2\pi\eta_s c_s^2}\, \Psi_2(\eta_s, \varphi), \tag{70}$$

where

$$\Psi_2(\eta_s, \varphi) = \sum_{n=0,2,\ldots} \sum_{m=-n}^{n} \left(A^s_{mn} + \bar{A}^s_{mn} \right) S_{mn}(-ic_s, \eta_s) e^{im\varphi}.$$

As was noted above, in order to find approximate values of the coefficients of the diffraction series determining the secondary field, for each m one must solve a truncated system of $2(N-|m|+1)$ equations with $2(N-|m|+1)$ unknowns. In the particular case of two equal disks, when the matrix elements of the system turn out to be connected by the relation

$$\alpha_{mnq}^{(-s,s)} = (-1)^{n+q}\alpha_{mnq}^{(s,-s)}, \qquad s = \pm 1,$$

the order of the finite systems to be solved can be reduced by half (for the same N), if one applies to them transformations analogous to those given at the end of [21].

5. It is not difficult to see that, in a similar way, the problem can also be solved in the case when the disks are excited by the field of an electric dipole located at an arbitrary point of space, provided that its moment is parallel to the surfaces of the disks. Indeed, then the electromagnetic field will be determined from (9), where the Hertz vector of the primary field is related to the dipole moment $\vec{p}$ by

$$\vec{\Pi} = \vec{p}\,\frac{e^{ikR}}{R} \tag{71}$$

(R is the distance from the dipole to the observation point), and if (71) is represented in the form of a series in spheroidal wave functions [6, 18], then the determination of the Hertz vector of the secondary field can be reduced to the problem considered above.

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