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Abstract

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ON THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS IN FOUR-DIMENSIONAL SPACE IN AN EXTENDED NEIGHBORHOOD OF A SADDLE-FOCUS

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Consider a system of 4 differential equations

$$dz/dt = Z(z), \tag{1}$$

where $Z(z)$ is an analytic vector-function. Suppose that system (1) has an equilibrium state O, at which the characteristic equation

$$|\partial Z/\partial z - \lambda E| = 0$$

has two pairs of complex conjugate roots $\lambda \pm i\omega,\ \gamma \pm i\Omega$, where $\lambda < 0,\ \gamma > 0$. Such an equilibrium state O will be called a saddle-focus. As is known, in this case system (1) has two two-dimensional integral manifolds $\mathfrak{M}^+$ and $\mathfrak{M}^-$, on which, respectively, the 0^+-curves and 0^--curves lie.

Suppose that there exists a trajectory Γ_0, leaving O and returning to it again as $t \to +\infty$, i.e. $\Gamma_0 \subset (\mathfrak{M}^+ \cap \mathfrak{M}^-)$. We shall assume that the indicated intersection is non-rough of first degree. This means that the relation

$$\dim(\mathcal{L}\mathfrak{M}^+_M \cap \mathcal{L}\mathfrak{M}^-_M) = 1, \tag{2}$$

holds, where by $\mathcal{L}\mathfrak{M}^+_M$ and $\mathcal{L}\mathfrak{M}^-_M$ are denoted the tangent spaces to $\mathfrak{M}^+_M$ and $\mathfrak{M}^-_M$ at the point $M \in \Gamma_0$.

Let $U(\Gamma_0)$ be some neighborhood of Γ_0. Obviously, $O \in U(\Gamma_0)$. The domain $U(\Gamma_0)$ will be called an extended neighborhood of the saddle-focus.

Theorem. *If $-\lambda \neq \gamma$ and condition (2) is fulfilled, then any extended neighborhood of the saddle-focus contains a countable set of periodic motions of saddle type.*

Without loss of generality one may assume that $\lambda + \gamma < 0$.

By means of a linear transformation, system (1) can be reduced to the form:

$$\begin{aligned} dx_1/dt &= \lambda x_1 - \omega x_2 + P, \qquad & dx_2/dt &= \omega x_1 + \lambda x_2 + P_2, \\ dy_1/dt &= \gamma y_1 - \Omega y_2 + Q, \qquad & dy_2/dt &= \Omega y_1 + \gamma y_2 + Q_2, \end{aligned} \tag{3}$$

where $O(0,0,0,0)$ is a saddle-focus; the functions $P_1(x_1,x_2,y_1,y_2),\dots,Q_2(x_1,x_2,y_1,y_2)$ vanish at the origin together with their first derivatives.

In a sufficiently small neighborhood of O, the equations of $\mathfrak{M}^+$ will be written in the form

$$y_1 = \varphi_1(x_1,x_2), \qquad y_2 = \varphi_2(x_1,x_2),$$

and the equations of $\mathfrak{M}^-$ in the form

$$x_1 = \psi_1(y_1,y_2), \qquad x_2 = \psi_2(y_1,y_2).$$

* The existence of a countable set of periodic motions in a neighborhood of Γ_0, leaving the saddle and returning to it as $t \to +\infty$, was discovered in works ([3],[4]) under the assumption that Γ_0 as $t \to -\infty$ enters the saddle, tangent to a certain one-dimensional axis.

Note that $\varphi_1,\dots,\psi_2$ are analytic functions, vanishing at the origin together with their first derivatives. After the substitution

$$\begin{aligned} \xi_1 &= x_1 - \psi_1(y_1,y_2), \qquad & \xi_2 &= x_2 - \psi_2(y_1,y_2), \\ \eta_1 &= y_1 - \varphi_1(x_1,x_2), \qquad & \eta_2 &= y_2 - \varphi(x_1,x_2), \end{aligned}$$

in some neighborhood O the system takes the form

$$\begin{aligned} d\xi_1/dt &= \lambda\xi_1 - \omega\xi_2 + P_{11}\xi_1 + P_{12}\xi_2, \\ d\xi_2/dt &= \omega\xi_1 + \lambda\xi_2 + P_{21}\xi_1 + P_{22}\xi_2, \\ d\eta_1/dt &= \gamma\eta_1 - \Omega\eta_2 + Q_{11}\eta_1 + Q_{12}\eta_2, \\ d\eta_2/dt &= \Omega\eta_1 + \gamma\eta_2 + Q_{21}\eta_1 + Q_{22}\eta_2, \end{aligned} \tag{4}$$

where $P_{ij}(0,\dots,0) = Q_{ij}(0,\dots,0) = 0$. In the new variables the equations of $\mathfrak{M}^+$ will be $\eta_1 = 0, \eta_2 = 0$, and those of $\mathfrak{M}^-$ will be $\xi_1 = 0, \xi_2 = 0$.

Denote by S_0 the surface $\xi_1^2 + \xi_2^2 = r_2,\ \eta_1^2 + \eta_2^2 \le r^2$, and by S_1 the surface $\eta_1^2 + \eta_2^2 = r^2,\ \xi_1^2 + \xi_2^2 \le r^2$. From the form of equations (4) it readily follows that

Lemma 1. *For all sufficiently small $r > 0$, the surfaces S_0 and S_1 are surfaces without contact for the trajectories of system (4).*

Let

$$\xi_i(t) = \xi_i(t, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \qquad \eta_i(t) = \eta_i(t, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0)$$

be the equation of a trajectory l of system (4) passing through the point $M_0(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in S_0$ at $t = 0$, where $\eta_1^{0\,2} + \eta_2^{0\,2} \neq 0$. From Lemma 1 we obtain that l, for some t_0, intersects S_1 at the point $M_1(\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1)$, where $\xi_1^{1\,2} + \xi_2^{1\,2} < r^2$. We denote this mapping of S_0 into S_1 along trajectories by T_0. Evidently, it will be written in the form:

$$\xi_i^1 = \xi_i(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \qquad \eta_i^1 = \eta_i(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \tag{5}$$

where the transition time t_0 of the phase point from S_0 to S_1 is found from the equation

$$\eta_1^2(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) + \eta_2^2(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = r^2.$$

Along with this form of writing the mapping T, which was constructed in an analogous way in works (1–4), we shall here construct another representation of the mapping T_0, which we shall call parametric.

Consider the system of integral equations

$$\begin{aligned}
\xi_1(t) &= e^{\lambda t}\,[\,\xi_1^0 \cos\omega t - \xi_2^0 \sin\omega t\,] + \\
&\quad + \int_0^t e^{\lambda(t-\tau)}[\,\overline{P}_1 \cos\omega(t-\tau) - \overline{P}_2 \sin\omega(t-\tau)\,]\,d\tau, \\
\xi_2(t) &= e^{\lambda t}\,[\,\xi_1^0 \sin\omega t + \xi_2^0 \cos\omega t\,] + \\
&\quad + \int_0^t e^{\lambda(t-\tau)}[\,\overline{P}_1 \sin\omega(t-\tau) + \overline{P} \cos\omega(t-\tau)\,]\,d\tau, \\
\eta_1(t) &= e^{\gamma(t-t_0)}[\,\eta_1^1 \cos\Omega(t-t_0) - \eta_2^1 \sin\Omega(t-t_0)\,] + \\
&\quad + \int_{t_0}^t e^{\gamma(t-\tau)}[\,\overline{Q}_1 \cos\Omega(t-\tau) - \overline{Q}_2 \sin\Omega(t-\tau)\,]\,d\tau, \\
\eta_2(t) &= e^{\gamma(t-t_0)}[\,\eta_1^1 \sin\Omega(t-t_0) + \eta_2^1 \cos\Omega(t-t_0)\,] + \\
&\quad + \int_{t_0}^t e^{\gamma(t-\tau)}[\,\overline{Q}_1 \sin\Omega(t-\tau) + \overline{Q}_2 \cos\Omega(t-\tau)\,]\,d\tau,
\end{aligned} \tag{6}$$

where $\overline{P}_1$ denotes the nonlinear terms of the first equation of system (4), $\overline{P}_2$ those of the second, etc.

From the method of successive approximations it follows easily

Lemma 2. *There exists a sufficiently small neighborhood* Σ *of the origin of the coordinates such that, under the condition* $M_0(\xi_1^0, \xi_2^0, 0, 0) \in \Sigma$, $M_1(0, 0, \eta_1^1, \eta_2^1) \in \Sigma$, *the solution of system (6) for all* $0 \le t \le t_0$ *exists and is unique.*

By verification we are convinced that the solution found,

$$\begin{aligned} \xi_i(t) &= \xi_i^\Pi(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \\ \eta_i(t) &= \eta_i^\Pi(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \quad i = 1, 2, \end{aligned} \tag{7}$$

is a solution of system (4), satisfying the conditions

$$\xi_i(0) = \xi_i^0, \qquad \eta_i(t_0) = \eta_i^1, \qquad i = 1, 2.$$

Lemma 3. *The solution (7) is representable in the form:*

$$\begin{aligned} \xi_1(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{11}^0 + \beta_{11}^0)\cos\omega t - \xi_2^0(1 + \alpha_{12}^0 + \beta_{12}^0)\sin\omega t], \\ \xi_2(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{21}^0 + \beta_{21}^0)\sin\omega t + \xi_2^0(1 + \alpha_{22}^0 + \beta_{22}^0)\cos\omega t], \\ \eta_1(t) &= e^{\gamma(t-t_0)}[\eta_1^1(1 + \alpha_{11}^1 + \beta_{11}^1)\cos\Omega(t - t_0) - \\ &\qquad - \eta_2^1(1 + \alpha_{12}^1 + \beta_{12}^1)\sin\Omega(t - t_0)], \\ \eta_2(t) &= e^{\gamma(t-t_0)}[\eta_1^1(1 + \alpha_{21}^1 + \beta_{21}^1)\sin\Omega(t - t_0) + \\ &\qquad + \eta_2^1(1 + \alpha_{22}^1 + \beta_{22}^1)\cos\Omega(t - t_0)], \end{aligned} \tag{8}$$

where $\alpha_{ij}^0(t - t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\alpha_{ij}^1(t, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\beta_{ij}^k(t, t_0, \xi_1^0, \dots, \eta_2^1)$ are analytic functions satisfying the conditions: $\alpha_{ij}^k(0, \xi_1^0, \dots, \eta_2^1)$ vanish for $\xi_1^0 = \xi_2^0 = \eta_1^1 = \eta_2^1 = 0$, while $\beta_{ij}^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\beta_{ij}^1(0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$ tend to zero as $t_0 \to +\infty$ together with their first derivatives.

Introduce the functions

$$\xi_i^1 = \xi_i^\Pi(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \qquad \eta_i^0 = \eta_i^\Pi(0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1). \tag{9}$$

From lemma (3) there will follow the estimates

$$|\xi_1^1| + |\xi_2^1| < Ke^{\lambda t_0}, \qquad |\eta_1^0| + |\eta_2^0| < Ke^{-\gamma t_0}, \tag{10}$$

$$D\xi_1^1 + D\xi_2^1 < Ke^{\lambda \cdot 2\pi t_0}, \qquad D\eta_1^0 + D\eta_1^1 < Ke^{-\gamma \cdot 2\pi t_0}, \tag{11}$$

where $D = |\partial/\partial t_0| + \Sigma|\partial/\partial\xi_i^0| + \Sigma|\partial/\partial\eta_i^1|$, and K is a certain constant. Obviously, when ${\xi_1^0}^2 + {\xi_2^0}^2 = r^2$, ${\eta_1^1}^2 + {\eta_2^1}^2 = r^2$, where r is sufficiently small, formulas (9) give a new form for writing the map T_0.

Without loss of generality one may assume r to be such that Γ_0 intersects S_0 and S_1 at the points $M_0^+(r,0,0,0)$ and $M_1^-(0,0,r,0)$. From the theorems on continuous dependence on initial conditions it follows that the trajectory of system (4) intersecting S_1 at points close to the point M_1^- will also intersect S_0 at points close to the point M_1^-. In the variables under consideration this correspondence, which we denote by T_1, can be written in a sufficiently small neighborhood of M_1^- in the form:

$$\bar{\eta}_i^0 = f(\xi_1^1, \xi_2^1, \eta_2^1) = A_1\xi_1^1 + A_2\xi_2^1 + B\eta_2^1 + \dots,$$

$$\bar{\xi}_i^0 = g_i(\xi_1^1, \xi_2^1, \eta_2^1) = A_{i1}\xi_1^1 + A_{i2}\xi_2^1 + B_i\eta_2^1 + \dots, \qquad i = 1, 2. \tag{12}$$

In the linear approximation, the trace of the intersection of $\mathfrak{M}^-$ with S_0 has the equations

$$\xi_2^0 = B\eta_2^1, \qquad \eta_1^0 = B_1\eta_2^1, \qquad \eta_2^0 = B_2\eta_2^1. \tag{13}$$

It follows from condition (2) that B_1 and B_2 are not simultaneously equal to zero.

Since, for sufficiently small $\bar{\eta}_1^0$ and $\bar{\eta}_2^0$, the trajectory passing through the point $\overline{M}_0(\bar{\xi}_1, \bar{\xi}_2^0, \bar{\eta}_1^0, \bar{\eta}_2^1)$ intersects S_1 after time $\bar{t}_0$ at the point $\overline{M}_1(\bar{\xi}_1, \bar{\xi}_2^1, \bar{\eta}_1^1, \bar{\eta}_2^1)$, then, by virtue of what was indicated above, the relation between the last two coordinates of the points $\overline{M}_0$ and $\overline{M}_1 = T\overline{M}_0$ can also be written in the form:

$$\bar{\eta}_i^0 = \eta_i^{\mathrm{II}}(0, \bar{t}_0, \bar{\xi}_1^0, \bar{\xi}_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1),$$

where $\bar{\xi}_1^0 = \sqrt{r^2 - (\bar{\xi}_2^0)^2}$, $\bar{\eta}_1^1 = \sqrt{r^2 - (\bar{\eta}_2^1)^2}$. Consider the mapping $T = T_1 T_0$

$$\begin{gathered} \xi_2^0 = f(\xi_1^{\mathrm{II}}(t_0, t_0, \xi_1^0, \dots, \eta_2^1), \xi_2^{\mathrm{II}}(t_0, t_0, \xi_1^0, \dots, \eta_2^1), \eta_2^1), \\ \eta_i^{\mathrm{II}}(0, t_0, \xi_1^0, \dots, \eta_2^1) = g_i(\xi_1^{\mathrm{II}}(t_0, t_0, \xi_1^0, \dots, \eta_2^1), \xi_2^{\mathrm{II}}(t_0, t_0, \xi_1^0, \dots, \eta_2^1)\eta_2^1),. \end{gathered} \tag{14}$$

where $\xi_1^0 = \sqrt{r^2 - (\xi_2^0)^2}$, $\eta_1^1 = \sqrt{r^2 - (\eta_2^1)^2}$. Obviously, T is defined for sufficiently small ξ_2^0, η_2^1 and sufficiently large t_0, and maps the "point" (ξ_2^0, η_2^1, t_0) into the point $(\bar{\xi}_2^0, \bar{\eta}_2^1, \bar{t}_0)$. We shall look for fixed points of T. The coordinates of the fixed points (ξ_2^*, η_2^*, t^*) satisfy the equations

$$\xi_2^* - f(\xi_1^{\mathrm{II}}, \xi_2^{\mathrm{II}}, \eta_2^*) = 0, \qquad \eta_i^{\mathrm{II}} - g_i(\xi_1^{\mathrm{II}}, \xi_2^{\mathrm{II}}, \eta_2^*) = 0, \qquad i = 1, 2. \tag{15}$$

Let, for definiteness, $B_1 \neq 0$. It is easily proved that the first two equations of system (15) are solvable with respect to ξ_2^* and η_2^* for all $t^* > \bar{t}^*$. Substituting their expressions into the last equation, we obtain an equation with respect to t^*

$$\begin{aligned} &B_1(1-\varphi_{21}^1(0,r,0,r,0)+\ldots)\sin\Omega t^* + \\ &\quad + B_2(1+\varphi_{11}^1(0,r,0,r,0)+\ldots)\cos\Omega t^* - \\ &\quad - e^{(\lambda+\gamma)t^*}\left[\Delta_1(1+\varphi_{11}^0(0,r,0,r,0)+\ldots)\cos\omega t^* + \right. \\ &\quad \left. + \Delta_2(1+\varphi_{21}^1(0,r,0,r,0)+\ldots)\sin\omega t^*\right] = 0, \end{aligned} \tag{16}$$

where $\Delta_1 = A_{11}B_2 - A_{21}B_1$, $\Delta_2 = A_{12}B_2 - A_{22}B_1$. Since $\lambda+\gamma < 0$ and $1+\varphi_{11}^1 > 0$, $1+\varphi_{21}^1 > 0$ for sufficiently small r, we obtain that equation (16) has a countable number of roots t_n^*, whose asymptotics as $n\to\infty$ is determined by the roots of the equation

$$B_1(1+\varphi_{21}^1)\sin\Omega t^* + B_2(1+\varphi_{11}^1)\cos\Omega t^* = \overline{B}\sin(\Omega t^*+\theta) = 0.$$

The characteristic equation of the linearized point mapping (14) at a fixed point is easy to find and can be written in the form

$$\alpha_1(n)z^3 + [\overline{B}\Omega\delta + \alpha_2(n)]z^2 + \alpha_3(n)z + \alpha_4(n) = 0,$$

where $\alpha_i(n)\to 0$ as $n\to\infty$, and δ is equal either to $+1$ or to -1. As $n\to+\infty$, this equation has one root tending to infinity and two tending to zero.

Thus, we have proved that the mapping T has a countable number of fixed points. Consequently, system (3) has a countable set of periodic solutions of saddle type, which pass through the points

$$M_n^*(\xi_{1n}^*, \xi_{2n}^*, \eta_1^{\mathrm{II}}(0, t^*, \xi_{1n}^*, \xi_{2n}^*, \eta_{1n}^*, \eta_{2n}^*), \eta_2^{\mathrm{II}}) \in S_0.$$

From the construction of the mappings T_0 and T_1 it follows that any extended neighborhood of the saddle-focus contains a countable set of periodic solutions of the indicated type.

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Note: Figure translations are in progress. See original paper for figures.