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An example of a composite stationary point

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS 1967, Vol. III, No. 3 BRIEF COMMUNICATIONS

An Example of a Complex Equilibrium Point

A. D. MYSHKIS, L. E. REIZIN

In the article [**?**], the question was raised regarding the possibility of constructing an example of an isolated equilibrium point that belongs to the limit set of some other trajectory, but does not serve as either a positive or negative limit set for any trajectory other than the point itself. In the present article, such an example is constructed. Consider the system of differential equations:

$$\frac{d\zeta}{dt} = P(\zeta, \eta, \zeta),$$

$$\frac{d\eta}{dt} = Q(\zeta, \eta, \zeta),$$

(1)

$$\frac{d\zeta}{dt} = R(\zeta, \eta, \zeta),$$

where

$P(\zeta, \eta, \zeta) = \ldots$, $Q(\zeta, \eta, \zeta) = \ldots$. The functions are defined as $P(\zeta, \eta, \zeta) = \sum(2\zeta_i \ldots)$ and $Q(\zeta, \eta, \zeta) = \sum(S_i \ldots)$, where the sums are taken over all integer values of k.

$f_k = f(\rho; 2^{-k}\zeta, 2^{-k}\eta)$,

£ (((6 cos - : (- l) 0, sin (*/36)), 3-2-W+i))–+ 2) , - 0 ,

$$\exp\left(2 - 2(b-a)^2(t-a)^{-1}(b-t)^{-1} + 1\right), \text{ if } a < t < b,$$

$$f(t; a, b) = \ldots, \text{ if } t \in (a, b),$$

$$g(t; a, b) = \int_a^t f(\tau; a, b)\, d\tau \Big/ \int_a^b f(\tau; a, b)\, d\tau,$$

ft a cp = £(a 2 /p 2 ; sin 2 (2^/9), 1/2),

P = (£ 2 + ^ 2 + C 2) 1 / 2 ,

As shown in [FIGURE:N], it is easy to see that at any given point, no more than two terms under the summation sign in expressions (2) are non-zero. Consequently, the functions $P(\xi, \eta, \zeta)$ and $Q(\xi, \eta, \zeta)$ possess continuous derivatives of all orders throughout the entire space. This ensures that the theorems regarding the existence and uniqueness of solutions for system (1) are applicable at every point in the space. Furthermore, it is straightforward to verify that this system possesses a unique equilibrium point at the origin $O(0, 0, 0)$.

In the region defined by $a^2/p^2 > 1/2$, system (1) transforms into the following system:

$$d\xi/dt = -\eta \exp(-1/p),$$

$$d\eta/dt = \xi \exp(-1/p). \quad (3)$$

d Vdt = 0.

An Example of a Single Complex Equilibrium Point: The trajectories in this region are circles centered on the axis. Within the cone $a(2\pi/9)$ for $\rho = 0$, the system takes the following form:

$$d\xi/dt = \xi \exp(-1/\rho),$$

$$d\eta/dt = \eta \exp(-1/\rho),$$

$$d\zeta/dt = \zeta \exp(-1/\rho),$$

That is, its trajectories intersect the sphere along the normals. The graphs of the functions f and g are provided. For $\rho = (z + 2)$, the system is expressed as:

$$d\xi/dt = ((-1)^k\eta - \xi(1-\zeta))\exp(-1/\rho),$$
$$d\eta/dt = (\xi\cos(\pi/18) - (-1)^k\eta\sin(\pi/18) + \xi(1-\zeta))\exp(-1/\rho),$$

$$d\zeta/dt = ((-1)^k\eta\sin(\pi/18) + \zeta\eta)\exp(-1/\rho),$$

From this, since $q = 1$ when $p \to 0$, we obtain $\frac{dq}{dt} = 2\frac{d\ln t}{dt}\frac{dq}{dt}\exp(1/p)$, where $\frac{dp}{dt}$ is non-zero only in specific regions. If we perform a change of variables,

$l_i = 5\cos(m\pi/18) - (-1)^k C\sin(m\pi/18)$,

$(-1)^k S\sin(\pi/18) + \cos(\pi/18)$, when $a = (\pi/36)$. The only region where the system trajectories can pass through the sphere is $1/2$. We shall show that between the spheres $\rho = 3\cdot 2^{k+1}$ and $\rho = 2^k$, the trajectories of the system are contained within circular cones with a common axis. If $c < \sin(\alpha)$, the system transforms into the following system within this region:

$\frac{d\ln d}{dt} = ((-m_{ij} + b_i g_j)f_t + \delta_{ij}h_j)\exp(-1/p)$,

$\frac{dM_i}{dt} = ((\xi_{ij} + \eta_j g_j)f_t + q_{ij})\exp(-1/p)$,

d k/dt = (C; ft/,- + C; A/) exp (- 1 / p) ,

$y = i$, if $3\cdot 2^{-(i+2)} < 2^{-i}$, and $j = i+1$, $(i+1)(i+2)$. Consequently, we obtain $a = \text{const}$. From points 2° and 3°, it follows that those trajectories passing through the sphere $r = 3\cdot 2^{-i}$ reach the sphere and fall into the region defined on it.

a? + ,/p<*sin*(*/36), (5)

A. D. MYSHKIS and L. E. REIZIN demonstrated that those trajectories passing through the sphere $\rho = 3\cdot 2^{-k}$ reach the sphere $\rho = 2^{-k}$ and enter a specific region upon it defined by

$a^2/\rho^2 < \sin^2(\pi/36)$. (6)

If there were trajectories that passed through both spheres, they would necessarily have to fall into both regions (5) and (6) simultaneously. However, this is impossible because the simultaneous fulfillment of both inequalities (5) and (6), in view of (4), leads to the contradictory inequality

$c^2\cos^2(\pi/18) + c^2\sin^2(\pi/18) < \rho^2\sin^2(\pi/36)$.

Thus, no trajectory can cross two spheres of the form $\rho = 3\cdot 2^{-k}$ in succession. [This

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arbitrary convex function in x^1, x^2 (i.e. satisfying condition (7)), possessing bounded first derivatives:

$$\left|\frac{\partial\phi(x^1, x^2, u)}{\partial x^1}\right| \leqslant M, \quad \left|\frac{\partial\phi(x^1, x^2, u)}{\partial x^2}\right| \leqslant M, \quad \left|\frac{\partial\phi(x^1, x^2, u)}{\partial u}\right| \leqslant M.$$

Then, letting

$$f(x^1, x^2, u) = -x^1 + u + \mu\phi(x^1, x^2, u),$$

we obtain for soctativnly small μ an object (3), satisfying all conditions (4) — (7).

Note also, that if the function f has the form

$$f(x^1, x^2, u) = -x^1 + g(x^2, u),$$

where g is a fynction satisfying the conditions

$$g(0, 1) > 0, \quad g(0, -1) < 0,$$

$$\left|\frac{\partial g(x^2, u)}{\partial x^2}\right| < 2, \quad \frac{\partial g(x^2, u)}{\partial u} > 0, \frac{\partial^2 g}{(\partial x^2)^2} \leqslant 0 \quad \text{for vell } x^2, u,$$

then all conditions (4) — (7) are also fulfillled (see [2], p. 510).

For the object under consideration, described by relations (2) — (7), we consider the problem of fastest arrival at the origin. As we shall see, the synthesis of optimal controls for the nonlinear object under consideration will be qualitatively the same as for the linear object $\ddot{x} + x = u$, i.e. the optimal trajectories will approach the origin, like spirals (see Fig. 13, p. 41 of book [1]) in Figure 10, presented below. It is interesting that such an "oscillatory" character of the optimal trajectories is connected with the fulfillment of condition (6), which can be called the condition of "strong negativity" of the derivative $\frac{\partial f}{\partial x^1}$. As shown in paper [3], replacing this condition with the condition of nc-negativity of the derivative $\frac{\partial f}{\partial x^1}$. leads leads to the fact that to the fact that each optimal control has only one switching, and the picture of the synthesis resembles the one we have for the linear object $\ddot{x} = u$ (see in book [1], p. 29 — 34 and, in particular, Fig. 7).

We proceed to the solution of the problem of optimal time-optimality for the object (2) — (7). First of all, we find all phase trajectories, satisfying the maximum principle ([1], p. 26). The function H has the following form — for the object under consideration:

$$H = \psi_1 x^2 + \psi_2 f(x^1, x^2, u). \tag{8}$$

We write further the system of equations for the auxiliary unknowns ψ_1, ψ_2:

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\partial H}{\partial x^1} = -\psi_2 \frac{\partial f}{\partial x^1}, \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x^2} = -\psi_1 - \psi_2 \frac{\partial f}{\partial x^2}. \end{aligned} \tag{9}$$

The maximum condition (along the optimal trajectory the function H achieves its maximum with respect to u) means in this case, that the maximum is acheved-

Figure 1: Figure 2

2. Distribution of various regions for system (1)

This demonstrates that no trajectory of system (1) can contain the origin in its limit set

. Finally, it should be noted that system (1) remains invariant under the transformation $\xi = -\xi'$, $\eta = -\eta'$, and $\zeta = -\zeta'$, as all functions P_i are odd with respect to these variables. In particular, the field directions at symmetric points are opposite. We now construct the system:

$$d\xi/dt = \sum_k P(\xi, \eta, \zeta - 2k\pi)F_k,$$

$$d\eta/dt = \sum_k Q(\xi, \eta, \zeta - 2k\pi)F_k, \quad d\zeta/dt = R(\xi, \eta, \zeta),$$

where the sums are taken over all integers k. The right-hand sides of system (7) possess continuous derivatives of all orders throughout the entire space; therefore, the existence and uniqueness of a trajectory passing through any point in space is guaranteed for system (7). System (7) is periodic with respect to ζ with a period of 2π. Indeed, if ζ is replaced by $\zeta + 2\pi$ and k is replaced by $k+1$, the form of the system remains unchanged. It follows from (8) that in the regions $-2\pi/3 < \zeta - 2k\pi < 2\pi/3$, each of the sums in (7) contains only one term, and the entire system (7) coincides with system (1). Consequently, it follows from Section 4 that the points $\zeta = \pi$ are isolated equilibrium points that do not belong to the limit sets of any other trajectories. System (7) possesses other isolated equilibrium points as well, specifically $\zeta = (2k + 1)\pi$. However, these other equilibrium points will not be of interest to us in the following discussion.

Among the properties of system (7), it should be noted that for large a, it takes the form (3). Finally, we construct the system:

$$d\xi/dt = (2\delta \sin(\zeta/2)(\sin(\zeta/2) + \cos(\zeta/2))H/(2a),$$

$$d\eta/dt = (2\delta \sin(\zeta/2)(\sin(\zeta/2) + \cos(\zeta/2)) - aH/(2a),$$

$$d\zeta/dt = \sum_k R(\xi, \eta, \zeta - 2k\pi)F_k,$$

(a, ξ, η) exp $(2\delta(\zeta/2))$ if $a > 2\delta \sin(\zeta/2)$, and $a < 2\delta \sin(\zeta/2)$.

Distribution of various regions for system (9). The right-hand sides of system (9), like those of system (7), have continuous partial derivatives of all orders throughout the space. For $|2\delta \sin(\zeta/2)|$ and for $a > 7$, system (9) coincides with system (7), and specifically for $a > 7$, it coincides with system (3). The surface

$\dot{\phi} = 2\zeta \sin^2(\zeta/2)$ (11)

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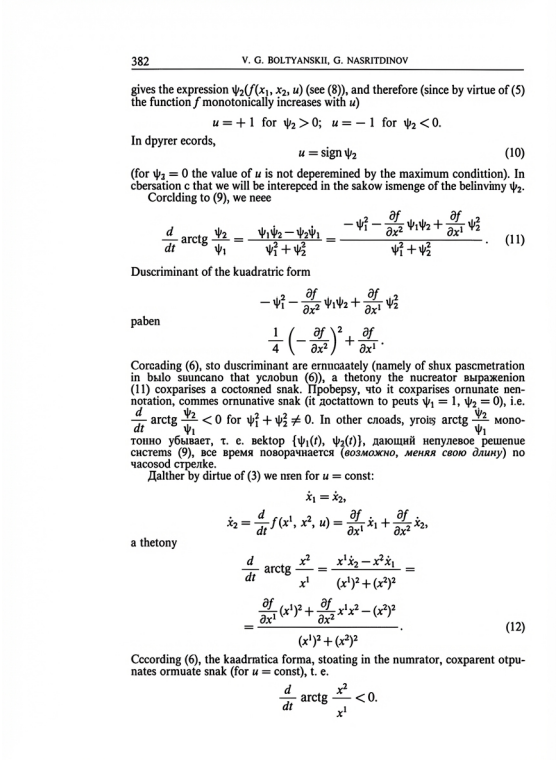

gives the expression $\psi_2(f(x_1, x_2, u)$ (see (8)), and therefore (since by virtue of (5) the function f monotonically increases with u)

$$u = +1 \text{ for } \psi_2 > 0; \quad u = -1 \text{ for } \psi_2 < 0.$$

In dpyrer ecords,

$$u = \operatorname{sign} \psi_2 \tag{10}$$

(for $\psi_2 = 0$ the value of u is not deperemined by the maximum condition). In cbersation c that we will be interepced in the sakow ismenge of the belinviny ψ_2.

Corclding to (9), we neee

$$\frac{d}{dt} \operatorname{arctg} \frac{\psi_2}{\psi_1} = \frac{\psi_1\dot{\psi}_2 - \psi_2\dot{\psi}_1}{\psi_1^2 + \psi_2^2} = \frac{-\psi_1^2 - \frac{\partial f}{\partial x^2}\psi_1\psi_2 + \frac{\partial f}{\partial x^1}\psi_2^2}{\psi_1^2 + \psi_2^2}. \tag{11}$$

Duscriminant of the kuadratric form

$$-\psi_1^2 - \frac{\partial f}{\partial x^2}\psi_1\psi_2 + \frac{\partial f}{\partial x^1}\psi_2^2$$

paben

$$\frac{1}{4}\left(-\frac{\partial f}{\partial x^2}\right)^2 + \frac{\partial f}{\partial x^1}.$$

Corcading (6), sto duscriminant are ernucaately (namely of shux pascmetration in bыlo suuncano that условuп (6)), a thetony the nucreator выражепion (11) coxparises a coctoяned snak. Пробевpsy, что it coxparises ornuate nen-notation, commes ornunative snak (it дoctattown to peuts $\psi_1 = 1$, $\psi_2 = 0$), i.e. $\frac{d}{dt} \operatorname{arctg} \frac{\psi_2}{\psi_1} < 0$ for $\psi_1^2 + \psi_2^2 \neq 0$. In other cloads, yroiиs $\operatorname{arctg} \frac{\psi_2}{\psi_1}$ mono-тонно убывает, т. е. вектор $\{\psi_1(t), \psi_2(t)\}$, дающий непулевое решепие cиcтems (9), все время поворачнается (*возможно, меняя свою длину*) no часosod стрелке.

Дalther by dirtue of (3) we пеп for $u = \text{const}$:

$$\dot{x}_1 = \dot{x}_2,$$

$$\dot{x}_2 = \frac{d}{dt} f(x^1, x^2, u) = \frac{\partial f}{\partial x^1}\dot{x}_1 + \frac{\partial f}{\partial x^2}\dot{x}_2,$$

a thetony

$$\frac{d}{dt} \operatorname{arctg} \frac{x^2}{x^1} = \frac{x^1\dot{x}_2 - x^2\dot{x}_1}{(x^1)^2 + (x^2)^2} =$$

$$= \frac{\frac{\partial f}{\partial x^1}(x^1)^2 + \frac{\partial f}{\partial x^2}x^1x^2 - (x^2)^2}{(x^1)^2 + (x^2)^2}. \tag{12}$$

Cccording (6), the kaadrиatica forma, stoating in the numrator, coxparent otpu-nates ormuate snak (for $u = \text{const}$), т. e.

$$\frac{d}{dt} \operatorname{arctg} \frac{x^2}{x^1} < 0.$$

Figure 2: Figure 3

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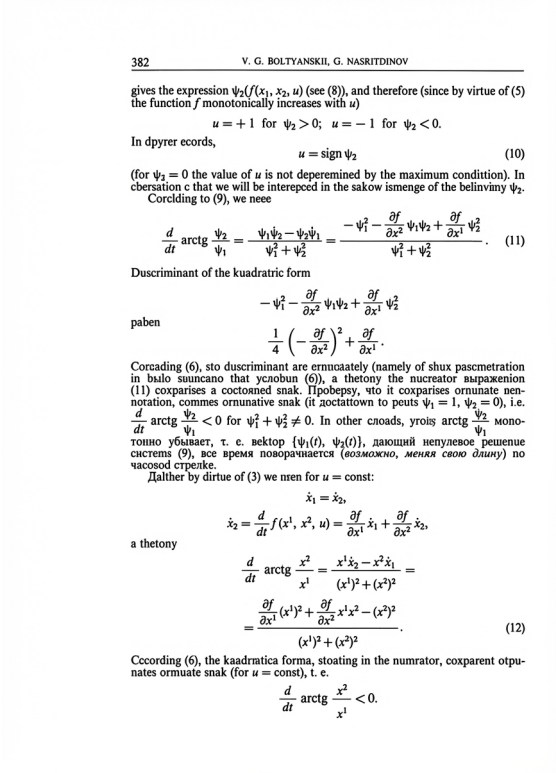
The phase space is filled with closed trajectories where const, which degenerate into points on the ζ-axis as they approach the limit. For $5|\sin(\zeta/2)| < \delta$, we have $\dot{\zeta} = 2\zeta\sin(\zeta/2)\cos(\zeta/2)$. In this equation, $\alpha(\zeta/2)$ serves as the solution toward which all other solutions converge as $\zeta \to \phi$



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$$\frac{d}{dt}\operatorname{arctg}\frac{x^2}{x^1} = \frac{x^1\dot{x}_2 - x^2\dot{x}_1}{(x^1)^2 + (x^2)^2} =$$

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$$\frac{d}{dt}\operatorname{arctg}\frac{x^2}{x^1} < 0.$$

Figure 3: Figure 3

. Indeed, if we denote $(\zeta/2) = ...$

$= |A|$, then

$(-1/\text{fx})$, d-Udt = exp $(-1/\text{p.})$

dfx/dC = $-$

From equation (12), it is evident that as $C \to \infty$, the value of ϕ also approaches infinity, although the rate of increase decreases as C grows. As $t \to \infty$, the variable μ increases and eventually exceeds the limit $36 - 25(C/2)$. It is easy to see that while $C \to \infty$, the term μ tends toward a finite limit by virtue of the expression for the factor in (10).

DIFFERENTIAL EQUATIONS A. D. MYSHKIS, L. E. REIZIN Let us define the mapping:

$$x = \left(8 + \phi + (S^2 + 16)^{1/2}\right) \cos C$$

$$y = \left(8 + \phi + (S^2 + 16)^{1/2}\right) \sin C$$

The inverse mapping is given by:

g = (r - 8) / (2 r) 1 / 2 ,

$z = \zeta/(2\Gamma)^{1/2}$

$\arctan(y/x) = \phi$. The general equilibrium points are $(0, 0, 0)$, from which it is evident that the mapping is topological and even analytic in both directions, provided that in the (x, y, z) space we restrict ourselves to the domain $x^2 + y^2 > 0, -\infty < z < +\infty$, and in the (ξ, η, ϕ) space we identify points with coordinates ϕ and $\phi + 2\pi$. Due to the periodicity of the right-hand sides of system (9) with respect to ϕ, the trajectories of the system will transform into the trajectories of the system obtained from (9) after substitution (13). For $a > \gamma$ (i.e., outside the region S), the system after mapping (13), according to Section 7, yields

$dx/dt = -2xz(z + 8)^{-1}$

- dy/dt = –2yz(r^8)- 1 ,

$$\frac{dz}{dt} = -\delta - \phi z - z^2 \frac{z + \delta + (2z)^{1/2}}{2z(z + \delta)}$$

From this, it is evident that for $z > 0$, the right-hand sides of the derivatives are continuous. It would have been possible to maintain the continuity of the derivatives of all orders by multiplying the right-hand sides by $\exp(-1/z)$; however, this is not essential for our purposes. Under the mapping (13), the ζ-axis is transformed into the circle $x^2 + y^2 = \delta^2$, and the equilibrium point O is mapped to the equilibrium point $x = \delta, y = 0$.

0. All

Trajectories passing through the points of the circle $\theta = 0, -a < r < a$ (excluding the point $r = 0, \theta = 0$), according to Section 8°, have the surface defined by Eq. (11) as their positive limit set.

Specifically, this includes the point $x = 0$. Thus, while the point belongs to the positive limit set of other trajectories, it does not constitute the entire limit set for any trajectory other than itself

. This system possesses properties similar to the system constructed by Zaremba in his article, yet it maintains a simpler structure.

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In other words, on each piece of the solution $\{x^1(t), x^2(t)\}$ of system (3), where $u = \text{const}$, the phase velocity vector $\{x^1(t), x^2(t)\}$ always rotates clockwise.

Let $x_0 = (x_0^1, x_0^2)$ be an arbitrary point of the phase plane, for which $\{x_0^2, f(x_0^1, x_0^2, 1)\} \neq 0$. Let us consider the solution $x(t)$, $\psi(t)$ of system (3), (9) for $u = 1$, satisfying the initial conditions

$$x^1(0) = x_0^1, \quad x^2(0) = x_0^2, \quad \psi_1(0) = 1, \quad \psi_2(0) = 0. \tag{13}$$

By virtue of (13), we find from the second equation (9) that $\dot{\psi}_2(0) = -1$. Consequently, $\psi_2(t) > 0$ for sufficiently small negative t. Let us denote by $T_+(x_0)$ the closest to zero negative root of the function $\psi_2(t)$ (if $\psi_2(t)$ does not vanish for $t < 0$, then we set $T_+(x_0) = m_1$, where (m_1, m_2) is the maximum interval of existence of the solution $x(t)$, $\psi(t)$; in particular, it may turn out that $T_+(x_0) = -\infty$). It is clear that on the interval $T_+(x_0) < t < 0$ the function $\psi_2(t)$ maintains a positive sign. The piece of the found trajectory $x(t)$, corresponding to the time interval $T_+(x_0) \leqslant t \leqslant 0$, let us denote by $K_+(x_0)$. This piece ends at the point x_0; its beginning (existing for $T_+(x_0) \neq m_1$) let us denote by $\xi_+(x_0)$.

Note that along the piece of the trajectory $K_+(x_0)$ we have $\psi_2(t) > 0$, $u \equiv 1$, i.e., condition (10) is satisfied. In other words, the piece $K_+(x_0)$ satisffies the maximum principle.

Note also that by virtue of the relation $\{x_0^2, f(x_0^1, x_0^2, 1)\} \neq 0$, the point x_0 is not an equilibrium position of system (3) for $u = 1$, and therefore along the entire arc $K_+(x_0)$ the phase velocity vector $\{x^2, f(x^1, x^2, 1)\}$ is distinct from zero. In particular, at those points of the arc $K_+(x_0)$ where $x^2 = 0$, the relation $f(x^1, x^2, 1) \neq 0$ is satisfied.

Similarly, using the initial conditions

$$x^1(0) = x_0^1, \quad x^2(0) = x_0^2, \quad \psi_1(0) = -1, \quad \psi_2(0) = 0, \tag{14}$$

we construct (for $\{x_0^2, f(x_0^1, x_0^2, -1)\} \neq 0$) the solution of system (3), (9) for $u \equiv -1$; the closest to zero negative root of the function $\psi_2(t)$ let us denote by $T_-(x_0)$ (if $\psi_2(t) \neq 0$ for $t < 0$, then $T_-(x_0) = m_1$, where (m_1, m_2) is the maximum interval of existence of the considered solution). The piece of the trajectory $x(t)$, corresponding to the time interval $T_-(x_0) \leqslant t \leqslant 0$, let us denote by $K_-(x_0)$, and the initial point of this piece by $\xi_-(x_0)$. This piece $K_-(x_0)$ also satissfies the maximum principle.

Let us now prove, that if the point x_0 liet on the axis x^1, then both tonks $\xi_+(x_0)$, $\xi_-(x_0)$ also leat let on the axis x^1, and $K_+(x_0)$ (and also $K_-(x_0)$) is a convex arc, lying wholly on one side of the axis x^1, and the tangents to this arc at its ends are parallel to the axis x^2 (Fig. 1, *a*); if however the point x_0 does not leat on the axis x^1, then the tonks x_0 and $\xi_+(x_0)$ (and also x_0 and $\xi_-(x_0)$) liet on different sides of the axis x^1, and $K_+(x_0)$ (respectively way $K_-(x_0)$) is a convex arc, intersecting the axis x^1 ome 1 once (Fig. 1, *b*).

Indeed, along the arc $K_+(x_0)$ we have

$$H = \psi_1 x^2 + \psi_2 f(x^1, x^2, 1) = \text{const}. \tag{15}$$

Since, in addition, $\psi_2(0) = \psi_2(T_+(x_0)) = 0$, it follows that

$$\psi_1(0)x^2(0) = \psi_1(T_+(x_0))x^2(T_+(x_0)).$$

Figure 4: Figure 4

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References

Myshskis, A. D., and Reizin, L. E. *Izvestiya Akademii Nauk Latviyskoy SSR, Seriya Fizicheskikh i Tekhnicheskikh Nauk*, 80–86, S. K. Ann. Soc. Pol.

Math., (1937) 1938, Received by the editorial office

February 16th

1966, Physicotechnical Institute of Low Temperatures of the Ukrainian Academy of Sciences; Institute of Physics of the Latvian Academy of Sciences

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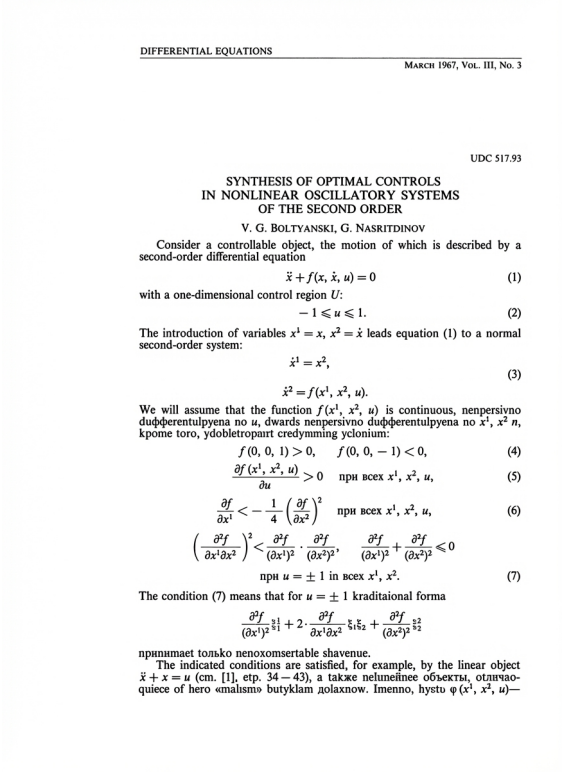
Figures



DIFFERENTIAL EQUATIONS

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SYNTHESIS OF OPTIMAL CONTROLS IN NONLINEAR OSCILLATORY SYSTEMS OF THE SECOND ORDER

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Consider a controllable object, the motion of which is described by a second-order differential equation

$$\ddot{x} + f(x, \dot{x}, u) = 0 \quad (1)$$

with a one-dimensional control region U:

$$-1 \leqslant u \leqslant 1. \quad (2)$$

The introduction of variables $x^1 = x$, $x^2 = \dot{x}$ leads equation (1) to a normal second-order system:

$$\dot{x}^1 = x^2, \qquad \dot{x}^2 = f(x^1, x^2, u). \quad (3)$$

We will assume that the function $f(x^1, x^2, u)$ is continuous, nenpersivno duффerentulpyena no u, dwards nenpersivno duффerentulpyena no x^1, x^2 n, kpome toro, ydobletropaırt credymming yclonium:

$$f(0, 0, 1) > 0, \qquad f(0, 0, -1) < 0, \quad (4)$$

$$\frac{\partial f(x^1, x^2, u)}{\partial u} > 0 \quad \text{при всех } x^1, x^2, u, \quad (5)$$

$$\frac{\partial f}{\partial x^1} < -\frac{1}{4}\left(\frac{\partial f}{\partial x^2}\right)^2 \quad \text{при всех } x^1, x^2, u, \quad (6)$$

$$\left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2 < \frac{\partial^2 f}{(\partial x^1)^2} \cdot \frac{\partial^2 f}{(\partial x^2)^2}, \qquad \frac{\partial^2 f}{(\partial x^1)^2} + \frac{\partial^2 f}{(\partial x^2)^2} \leqslant 0$$

при $u = \pm 1$ in всех x^1, x^2. (7)

The condition (7) means that for $u = \pm 1$ kraditaional forma

$$\frac{\partial^2 f}{(\partial x^1)^2}\xi_1^2 + 2\cdot\frac{\partial^2 f}{\partial x^1 \partial x^2}\xi_1\xi_2 + \frac{\partial^2 f}{(\partial x^2)^2}\xi_2^2$$

принимает только nenoxomsertable shavenue.

The indicated conditions are satisfied, for example, by the linear object $\ddot{x} + x = u$ (cm. [1], etp. 34 – 43), a также nelumeйnee объекты, отличаoquiece of hero «malısm» butyklam дolaxnow. Imenno, hystu $\varphi(x^1, x^2, u)$—

Figure 5: Figure 1

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But the numbers $\psi_1(0)$ and $\psi_1(T_+(x_0))$ have opposite signs, since the vector $\psi(t)$ rotates clockwise, and 0 and $T_+(x_0)$ are two adjacent zeros of the function $\psi_2(t)$ (Fig. 2). Consequently, the numbers $x^2(0) = x_0^2$ and $x^2(T_+(x_0))$ also have opposite signs, i.e., either both points x_0, $\xi_+(x_0)$ lie on the x^1 axis, or they lie on opposite sides of the x^1 axis.

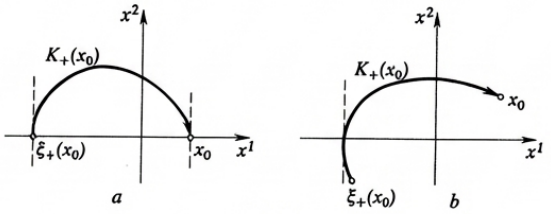

Fig. 1

In the first case (both points x_0, $\xi_+(x_0)$ lie on the x^1 axis), the constant (15) is equal to zero, since $x^2(0) = \psi_2(0) = 0$. If some intermediate point $x(\tau)$, $T_+(x_0) < \tau < 0$ of the arc $K_+(x_0)$ lay on the x^1 axis, then from the relations

$$H = \psi_1(\tau)x^2(\tau) + \psi_2(\tau)f(x^1(\tau), x^2(\tau), 1) = 0, \quad x^2(\tau) = 0, \quad f(x^1(\tau), x^2(\tau), 1) \neq 0$$

we would get $\psi_2(\tau) = 0$. But this contradicts the fact that $T_+(x)$ is the negative root of the function $\psi_2(t)$ closest to zero. Consequently,

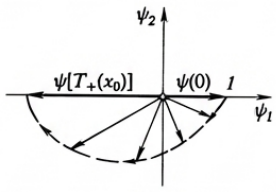

Fig. 2

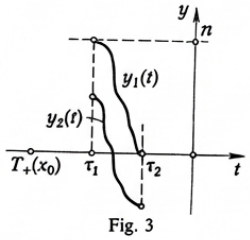

Fig. 3

the arc $K_+(x_0)$ has no common points with the x^1 axis, except for the ends, i.e., it is located entirely on one side of the x^1 axis. The convexity of this arc follows from the fact that the tangent vector $\{\dot{x}^1, \dot{x}^2\}$ of this arc turns all the time in one direction (clockwise). Since, finally, $x^1 = x^2 = 0$ at the endpoints of the arc $K_+(x_0)$, and $\dot{x}^2 = f(x^1, x^2, 1) \neq 0$, then at the endpoints of the arc $K_+(x_0)$ the tangent is parallel to the x^2 axis.

Let now the points x_0 and $\xi_+(x_0)$ lie on opposite sides of the x^1 axis. Then the arc $K_+(x_0)$ connecting them must necessarily intersect the x^1 axis. We will show that it can intersect the x^1 axis only once. Indeed, at each point of intersection of the arc $K_+(x_0)$ with the x^1 axis, we have $f(x^1, x^2, 1) \neq 0$, i.e., at this point the tangent vector $\{\dot{x}^1, \dot{x}^2\} = \{x^2, f(x^1, x^2, 1)\}$ is non-zero and is parallel to the x^2 axis (Fig. 1, *b*). Therefore, it is sufficient for us

Figure 6: Figure 5