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SPLITTING OPERATOR
FOR SYSTEMS OF
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 f)**

MATHEMATICS

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Abstract

Full Text

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MATHEMATICS

E. G. DYAKONOV

DIFFERENCE SCHEMES WITH A SPLITTING OPERATOR FOR SYSTEMS OF EQUATIONS OF THE FORM $L_0 \frac{\partial u}{\partial t} + L_1 u = f$

(Presented by Academician A. N. Tikhonov on November 22, 1966)

1. Let Ω be a domain in the space $x' = (x_1, x_2, \dots, x_p)$, composed of a finite number of parallelepipeds with faces parallel to the coordinate planes, and let in $Q_T \equiv \Omega \times [0 \leq x_0 \leq T]$ one seek the solution of the system of equations

$$L_0 D_0 u + L_1 u = f(x), \quad (1)$$

satisfying the initial and boundary conditions

$$u|_{x_0=0} = \varphi(x'); \quad D^\alpha u = 0 \text{ for } x' \in \Gamma \text{ and } |\alpha| \leq m = \max(m_0, m_1), \quad (2)$$

where $u \equiv (u_1(x), \dots, u_N(x))$; $x = (x_0, x')$; $D_s = \partial/\partial x_s$; $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_p^{\alpha_p}$; $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$; $|\alpha| = \sum_{s=1}^p \alpha_s$; $a_k^{\alpha\beta}(x), a_k^\alpha(x)$ are bounded matrices of order N ; Γ is the boundary of Ω ;

$$L_k \equiv \sum_{\substack{|\alpha| \leq m_k \\ |\beta| \leq m_k}} D^\alpha (a_k^{\alpha\beta}(x) D^\beta) + \sum_{|\alpha| \leq m_k} a_k^\alpha(x) D^\alpha \quad (k = 0, 1) \quad (3)$$

are strongly elliptic operators (1).

Let $h_s > 0$, $\tau > 0$ be the mesh steps with respect to x_s ($s = 1, 2, \dots, p$) and x_0 , respectively: $i = (i_1, i_2, \dots, i_p)$; $x_i = ih$; $v_i^n = v(n\tau, x_i)$; $\bar{\Omega}_h \equiv \{x_i : x_i \in \bar{\Omega}\}$; $\Omega_h \equiv \{x_i : x_{i+j} \in \bar{\Omega} \text{ for } |j| \leq m\}$; $\Gamma_h = \bar{\Omega}_h \setminus \Omega_h$; $\Delta^\alpha v$, Δv , $v_{x_0}^n$, $[v, w]$, $\|v\|$, $\|v\|_{W_2^m}$ are the notations from [2]; $A_{k,s} \equiv \sum_{r \leq m_k} (-1)^r \Delta_s^r (b_{k,s,r}(x_s) \cdot \Delta_s^r)$, $s = 1, 2, \dots, p$; $k = 0, 1$; $b_{k,s,r}(x_s)$ are

diagonal nonnegative matrices of order N ; A_k is the difference approximation obtained from (3) by replacing D^α by Δ^α and D^β by Δ^β ;

$$\hat{\Lambda}_0 \equiv \prod_{s=1}^p \Lambda_{0,s}; \quad \hat{\Lambda}_1 \equiv \sum_{r=1}^p \Lambda_{0,1} \dots \Lambda_{0,r-1} \Lambda_{1,r} \Lambda_{0,r+1} \dots \Lambda_{0,p}.$$

Consider the difference scheme*

$$\prod_{s=1}^p (\Lambda_{0,s} + \tau \Lambda_{1,s}) v_{x_0}^n + (\Lambda_0 - \hat{\Lambda}_0) v_{x_0}^{n-1} + \Lambda_1 v^n = f^n, \quad x_i \in \Omega_h; \quad (4)$$

$$v^0 = \varphi; \quad v^1 = \varphi^1; \quad v_i^n = 0 \text{ for } x_i \in \Gamma_h, \quad (5)$$

* Scheme (4), (5) was presented in a communication by the author at the International Mathematical Congress in Moscow in August 1966.

where all coefficients in (4) are computed at $x_0 = n\tau$, while $\varphi^1 \approx u^1$ and can be found, for example, by means of two-level difference schemes. To formulate the stability theorem we introduce the notation $\omega_l \equiv \{\alpha : \alpha_{s_1}, \alpha_{s_2}, \dots, \alpha_{s_l} \leq m_1, \text{ and the remaining } \alpha_s \leq m_0\}$, $\|v\|_l^2 \equiv \sum_{\alpha \in \omega_l} \|\Delta^\alpha v\|^2$, and assume that

$$\frac{1}{2} \{ [\hat{\Lambda}_0^n v, v] + [\hat{\Lambda}_0^{n-1} w, w] \} + [(\Lambda_0^n - \hat{\Lambda}_0^{n-1}) v, w] \geq \delta_0 (\|v\|_{W_2^{m_0}}^2 + \|w\|_{W_2^{m_0}}^2); \quad (6)$$

$$\Lambda_1 = \bar{\Lambda}_1 + \sum_{\substack{|\alpha| \leq m_0 \\ |\beta| \leq m_1}} \bar{\Delta}^\alpha (c^{\alpha\beta}(x) \Delta^\beta), \quad (7)$$

where $c^{\alpha\beta}(x)$ are bounded, $[\bar{\Lambda}_1 v, w] = [v, \bar{\Lambda}_1 w]$,

$$[\hat{\Lambda}_1 v, v] \geq \delta_1 \|v\|_{W_2^{m_1}}^2, \quad \left| \left[\frac{\bar{\Lambda}_1^{n+1} - \bar{\Lambda}_1^n}{\tau} v, v \right] \right| \leq \delta \|v\|_{W_2^{m_1}}^2,$$

$$2[\hat{\Lambda}_1 v, v] - [\bar{\Lambda}_1 v, v] \geq \delta_2 \|v\|_1^2, \quad (8)$$

v, w are arbitrary grid vector-functions vanishing on Γ_h ; δ are constants independent of the grid; $\delta_0, \delta_1, \delta_2 > 0$.

Theorem 1. *Let conditions (6)–(8) be satisfied and let the vector-function v be a solution of (4), (5) with*

$$f \equiv \sum_{|\alpha| \leq m} \bar{\Delta}^\alpha f_\alpha + \tau \hat{\Lambda}_0 f_0 + \tau \hat{\Lambda}_1 f_1 + \left\{ \prod_{s=1}^p (\Lambda_{0,s} + \tau \Lambda_{1,s}) - \hat{\Lambda}_0 - \tau \hat{\Lambda}_1 \right\} f_2. \quad (9)$$

Then for $k \leq T/\tau$ the estimate holds

$$\begin{aligned} V_k^1(v) &\equiv \tau \sum_{n=0}^{k-1} \|v_{x_0}^n\|_{W_2^{m_0}}^2 + \|v^k\|_{W_2^{m_1}}^2 + \sum_{n=1}^{k-1} \sum_{l=1}^p \tau^{l+1} \|v_{x_0}^n\|_l^2 \leq \\ &\leq M \left\{ \tau \|v_{x_0}^0\|_0^2 + \|v^1\|_{W_2^{m_1}}^2 + \sum_{n=1}^{k-1} \left[\sum_{l=2}^p \tau^{l+1} \|f_2^n\|_l^2 + \tau^2 \|f_1^n\|_1^2 \right] + \|f\|_f^2 \right\}, \quad (10) \end{aligned}$$

where

$$\begin{aligned} \|f\|_f^2 &\equiv \tau \sum_{n=1}^{k-1} \sum_{|\alpha| \leq m_0} \|f_\alpha^n\|^2 + \sum_{m_0 \leq |\alpha| \leq m} \left\{ \tau \sum_{n=1}^{k-2} \|(f_\alpha^n)_{x_0}\|^2 + \|f_\alpha^1\|^2 \right\} + \tau \sum_{n=1}^{k-1} \tau \|f_0^n\|_0^2 \quad \text{for } m_0 \leq m_1, \\ \|f\|_f^2 &\equiv \tau \sum_{n=1}^{k-1} \left(\sum_{|\alpha| \leq m_1} \|f_\alpha^n\|^2 + \tau^2 \|\hat{\Lambda}_0 f_0^n\|^2 \right) \quad \text{for } m_0 > m_1, \end{aligned}$$

and the constant M does not depend on τ, h_s, k .

Estimate (10) is related to the results ^(3,4) and remains valid when the operators $\Lambda_{k,s}$ are chosen from a more general class (see ^(2,3)).

Condition (6) is satisfied, for example, when

$$\Lambda_0^n = \bar{\Lambda}_0^n + R_0^n, \quad [\bar{\Lambda}_0^n v, w] = [v, \bar{\Lambda}_0^n w], \quad [\bar{\Lambda}_0^n v, v] \geq 2\delta_0 \|v\|_{W_2^{m_0}}^2, \quad (6')$$

$$|[R_0^n v, w]| \leq \delta_0 (\|v\|_{W_2^{m_0}}^2 + \|w\|_{W_2^{m_0}}^2), \quad [\bar{\Lambda}_0^n v, v] \leq [\hat{\Lambda}_0 v, v].$$

An important example of (1), for which Theorem 1 is valid, is the equation

$$-\Delta D_0 u + \nu \Delta^2 u = f, \quad \nu > 0.$$

It is not difficult to see that the theorem is easily generalized also to the case when, instead of $f(x)$, one has $f(x, D^\alpha u)$, $|\alpha| \leq m$, and the nonlinearity in $D^\alpha u$ is weak (see, for example, ⁽³⁾).

2. Let $a(x) \in H_\alpha(0)$ mean that in Q_T , $a(x)$ has continuous derivatives $D^\beta a$ ($\beta \leq \alpha$) satisfying the Lipschitz condition in all x_s whose indices coincide with the indices of the nonzero components of α . We also assume that $h_s = h_{\chi_s}$, where $0 < \underline{\chi} \leq \chi_s \leq \bar{\chi}$ and $\underline{\chi}, \bar{\chi}$ do not depend on the mesh.

Theorem 2. Let the conditions of Theorem 1 be satisfied, $a_k^{\alpha\beta} \in H_\alpha(0)$, $b_{k,s,r} \in H_s(r)$ (see (2)), $m_0 \leq m_1$, and let the solution of (1), (2) exist and possess in Q_T continuous derivatives $D^\alpha u$, $|\alpha| \leq 2m + 1$; $D^\alpha D_0^{\alpha_0} u$, $\alpha_0 \leq 2$, $\alpha \in \omega_0$, $|\alpha| \leq m_0(2p - 1)$; $D^\alpha D_0^{\alpha_0} u$, $\alpha_0 \leq 1$, $\alpha \in \omega_1$; $D^\alpha D_0^{\alpha_0} u$, $\alpha_0 \leq 1$, $\alpha \in \omega_1$, $|\alpha| \leq m_1 l + m_0(p - l - 1)$, $l = 2, \dots, p$; $D^\alpha D_0 u$, $|\alpha| \leq mp$. Then for $z \equiv u - v$ the estimate is valid

$$\{V_h^k(z)\}^{1/2} \leq M \left\{ \tau + h^{1/2} + \tau^{1/2} \|z_{x_0}^0\|_0 + \|z^1\|_{W_2^{m_1}} \right\}; \quad (11)$$

moreover, if $m \leq 1$, $\Gamma_h \in \Gamma$, then in (11) one may replace $h^{1/2}$ by h .

Remark 1. Estimate (11) remains valid also in the case when there are discontinuities of $a^{\alpha\alpha}$, $|\alpha| = \alpha_s$, on a finite number of smooth surfaces $x_s = x_s(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_p)$, under the condition that between the indicated surfaces $a^{\alpha\alpha}$ has the same derivatives as were required throughout Q_T by the notation $a^{\alpha\alpha} \in H_\alpha(0)$. Naturally, in this case, on the surfaces of discontinuity it is required, instead of (1), that $D^\beta u$, $\beta \leq \alpha$, and $a^{\alpha\alpha} D^\alpha u$ be continuous.

Remark 2. The quantities $\tau^{1/2} \|z_{x_0}^0\|_0$ and $\|z^1\|_{W_2^{m_1}}$ in (11) may be replaced by $R_0 \equiv \tau^{1/2} \|\hat{\Lambda}_0 D_0 u|_{x=0} - \psi(x')\|$, if $v^1 \equiv \varphi^1$ is found from

$$\prod_{s=1}^p (\Lambda_{0,s} + \tau \Lambda_{1,s}) v_{x_0}^0 + (\Lambda_0 - \hat{\Lambda}_0) \psi(x') + \Lambda_1 v^0 = f^0, \quad x_i \in \Omega_h. \quad (4')$$

Here, for $\hat{\Lambda}_0 D_0 u|_{x_0=0}$, one can sometimes find an approximation $\psi(x')$ such that $R_0 = O(\tau^{1/2} h)$. This can be done, for example, when from (1) for $x_0 = 0$ and (2) one can obtain a boundary-value problem for $w(x') \equiv \tilde{L}_0 D_0 u|_{x_0=0}$; the solution of this problem by the mesh method may be taken as $\psi(x')$, if $w(x')$ is sufficiently smooth.

Remark 3. For the stationary problem $L_1 \varphi = f(x')$; $D^\alpha \varphi|_\Gamma = 0$, $|\alpha| \leq m_1$, under the condition $(\Lambda_1 v, v) \geq \delta \|v\|_{W_2^{m_1}}^2$, $\delta > 0$, it follows from the proof of Theorem 2, based on representing the approximation error of (1) and (2) in the form (10), that $\|\varphi - v\|_{W_2^{m_1}} = O(h^{1/2})$. A similar result for one equation with constant coefficients was obtained in (5).

3. For schemes (4), (5), in the case when Ω is a parallelepiped, one can also obtain a priori estimates of the solution, similar to the estimates from (6, 7). For example, multiplying (4) by $\tau \tilde{\Lambda}_1 v_{x_0}^n \equiv \tau (-1)^{m_1} \sum_{s=1}^p \Delta_s^{m_1} \Delta_s^{m_2} v_{x_0}^n$ and summing over n from 1 to $k - 1$, one can obtain the estimate

$$\begin{aligned} & \tau \sum_{n=1}^{k-1} \|v_{x_0}^n\|_{\Lambda_0, \tilde{\Lambda}_1}^2 + \|v^k\|_{\tilde{\Lambda}_1, \tilde{\Lambda}_1}^2 + \sum_{l=1}^p \tau^{l+1} \sum_{n=1}^{k-1} \|v_{x_0}^n\|_l^2 \leq \\ & \leq M \left\{ \tau \|v_{x_0}^0\|_{\Lambda_0, \tilde{\Lambda}_1}^2 + \|v^1\|_{\tilde{\Lambda}_1, \tilde{\Lambda}_1}^2 + \tau \sum_{n=1}^{k-1} \|f^n\|^2 \right\}, \end{aligned} \quad (12)$$

where $\|v\|_{\Lambda_0, \tilde{\Lambda}_1}^2$ denotes $\sum_{\alpha, \beta, \gamma} \|\Delta^{\alpha+\beta+\gamma} v\|^2$, the vectors α, β, γ being such that

$$|\alpha| = |\beta| = m_0 \quad \text{and} \quad a_0^{\alpha\beta} \neq 0, \quad |\gamma| = 2m_1 = \gamma_s, \quad a \|v\|_l^2 = \sum_{\alpha \in \tilde{\omega}_l} \|\Delta^\alpha v\|^2;$$

$\tilde{\omega}_l$ differs from ω_l only in that one of the components α is additionally permitted to be $\leq 2m_1$ instead of $\leq m_1$. For (12) to hold when $m_0 > m_1$, it is sufficient that the conditions

$$\begin{aligned} & 1/2 \{ [\Lambda_0^n z^n, \tilde{\Lambda}_1 z^n] + [\Lambda_0^{n-1} z^{n-1}, \tilde{\Lambda}_1 z^{n-1}] \} + [(\Lambda_0^n - \tilde{\Lambda}_0) z^{n-1}, \tilde{\Lambda}_1 z^n] \geq \\ & \geq \delta \{ \|z^n\|_{\Lambda_0, \tilde{\Lambda}_1}^2 + \|z^{n-1}\|_{\Lambda_0, \tilde{\Lambda}_1}^2 \}, \end{aligned} \quad (13)$$

$$[(2\tilde{\Lambda}_1 - \tilde{\Lambda}_1)z, \tilde{\Lambda}_1 z] \geq \delta \|z\|_1^2, \quad (14)$$

$$\left[\left\{ \prod_{s=1}^p (\Lambda_{0,s} + \tau \Lambda_{1,s}) - \tilde{\Lambda}_0 - \tau \tilde{\Lambda}_1 \right\} z, \tilde{\Lambda}_1 z \right] \geq \delta \sum_{l=2}^p \tau^l \|z\|_l^2. \quad (15)$$

In the case $m_0 \leq m_1$, in (12) the term

$$\tau \sum_{n=1}^{k-1} \|f^n\|^2$$

should be replaced by

$$\tau \sum_{n=1}^{k-1} \|f_{x_0}^n\|^2 + \|f^1\|^2$$

and one should assume (14), (15), and also that

$$\Lambda_1 = \bar{\Lambda}_1 + R_1,$$

where

$$\begin{aligned} & [\tilde{\Lambda}_1 z, \tilde{\Lambda}_1 y] = [\tilde{\Lambda}_1 z, \bar{\Lambda}_1 y], \quad \left| [((\bar{\Lambda}_1^n - \tilde{\Lambda}_1^{n-1})z, \Lambda_1 z)] \right| \leq M \tau \|z\|_{\tilde{\Lambda}_1, \tilde{\Lambda}_1}^2, \quad [\bar{\Lambda}_1 z, \tilde{\Lambda}_1 z] \geq \\ & \geq \delta \|z\|_{\tilde{\Lambda}_1, \tilde{\Lambda}_1}^2, \quad [(2\tilde{\Lambda}_1 - \tilde{\Lambda}_1)z, \tilde{\Lambda}_1 z] \geq \delta \|z\|_1^2, \quad R_1 = \sum_{|\alpha| \leq 2m_0} r_\alpha(x) \Delta^\alpha \end{aligned}$$

and

$$|\Delta^q r_\alpha(x)| \leq M, \quad q \leq 2(m_1 - m_0).$$

Let us note, in conclusion, that there exists a whole class of a priori estimates similar to (12), obtained, for example, by multiplying (4) not by $\tau\tilde{\Lambda}_1 v_{x_0}^n$, but by

$$\tau\tilde{\Lambda}_0 v_{x_0}^n; \quad \tau\tilde{\Lambda}_1 v_{x_0}^n + \left\{ \prod_{s=1}^p (\Lambda_{0,s} + \tau\Lambda_{1,s}) - \hat{\Lambda}_0 - \tau\hat{\Lambda}_1 \right\} v_{x_0}^n,$$

and so on.

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

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