A LIMIT THEOREM FOR THE NUMBER OF CROSSINGS OF A HIGH LEVEL BY A STATIONARY GAUSSIAN PROCESS

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Abstract

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A LIMIT THEOREM FOR THE NUMBER OF CROSSINGS OF A HIGH LEVEL BY A STATIONARY GAUSSIAN PROCESS

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Let ξ_t be a stationary Gaussian process, $\mathbf{M}\xi_t = 0$, $\mathbf{M}\xi_s\xi_{s+t} = \rho(t)$, $-\infty < t < +\infty$. It is said that the process ξ_t crosses the level u from below upward at the time τ if $\xi_\tau = u$, and in a sufficiently small neighborhood $(\tau-\varepsilon, \tau+\varepsilon)$ of the point τ, $\varepsilon = \varepsilon(\omega)$ being a random variable, $\xi_s < u$, $\tau - \varepsilon < s < \tau$, $\xi_s > u$, $\tau < s < \tau + \varepsilon$. Denote by $\eta_u(\Delta)$ the number of crossings from below upward of the level u on the interval Δ. Denote by $\tilde{\eta}_u(\Delta)$ the accompanying random variable, taking integer values $\tilde{\eta}_u(\Delta) = \eta_u(\Delta)$ if in the interval Δ there are no crossings following one another after a time less than τ; otherwise we put $\tilde{\eta}_u(\Delta) = 0$.

Everywhere below it is assumed that the correlation function $\rho(t)$ is twice differentiable and satisfies the conditions $\rho(0) = 1$,

$$|\rho''(t) - \rho''(0)| \leq c/|\ln|t||^{1+\varepsilon}, \qquad c, \varepsilon > 0, \qquad t \to 0; \tag{1}$$

$$\rho(t) = o(1/\ln t), \qquad \rho'(t) = o(1/\sqrt{\ln t}), \qquad t \to \infty. \tag{2}$$

As the level u is increased $(u \to \infty)$, we shall increase $|\Delta|$—the length of the interval Δ—in such a way that $e^{-u^2/2}|\Delta| = \text{const}$. For brevity we denote such a coordinated increase of u and $|\Delta|$ by $(u, |\Delta|) \to \infty$. Denote by $A_u(|\Delta|, \tau)$ the event consisting in the fact that on the interval Δ there are crossings of the level u from below upward which follow one another after a time less than τ.

Theorem 1. *If the correlation function of the Gaussian process ξ_t satisfies (1), then for $\tau \leq v_u = \exp\{o(u^2)\}$, $u \to \infty$,*

$$\lim_{(u,|\Delta|)\to\infty} \mathbf{P}\{A_u(|\Delta|, \tau)\} = 0. \tag{3}$$

In the proof of the theorem the limiting relation

$$\lim_{u\to\infty}\frac{1}{\tau}e^{u^2/2}\int_0^{2\tau}\int_0^{2\tau}\mathbf{M}(\dot\xi^+_{t_1}\dot\xi^+_{t_2}\mid \xi_{t_j}=u,\ j=1,2)p_{t_1,t_2}(u,u)\,dt_1\,dt_2=0,$$

is used, where $p_{t_1t_2}(x_1,x_2)$ is the joint probability density of the values ξ_{t_1},ξ_{t_2}, and $\dot\xi^+_t=\dot\xi_t$ if $\dot\xi_t>0$; $\dot\xi^+_t=0$ if $\dot\xi_t\le 0$.

It follows from (3) that, for $(u,|\Delta|)\to\infty$, the limiting distributions of $\eta_u(\Delta)$ and $\tilde\eta_u(\Delta)$ coincide.

Consider the k-fold integrals defined by the formulas

$$J_{(k)}(\Delta;u,\tau)=\int_{\substack{t_i\in\Delta,\ |t_i-t_j|>\tau\\ i,j=1,\dots,k}}\cdots\int\mathbf{M}\left(\prod_{i=1}^k\dot\xi^+_{t_i}\,\middle|\,\xi_{t_j}=u,\ j=1,\dots,k\right)p_{t_1\dots t_k}(u\dots u)\,dt_1\dots dt_k. \tag{4}$$

It is easy to show that under the assumptions made $J_{(k)}(\Delta;u,\tau)<\infty$.
For the proof of the main assertions the following lemma is useful.

Lemma 1. Under a coordinated increase

$$(u,|\Delta|)\to\infty,\qquad |\Delta|=\frac{2\pi\mu}{\sqrt{-\rho''(0)}}e^{u^2/2}$$

there exist, for every k, numbers $\tau_k>0$ such that for all τ, $\tau_k<\tau<\sqrt{|\Delta|}$,

$$\lim_{(u,|\Delta|)\to\infty}J_{(k)}(\Delta;u,\tau)=\mu^k,\qquad k=1,2,\dots. \tag{5}$$

Let us now consider the behavior of the factorial moments $\widetilde J_k(\Delta,u)$ for the random variables $\tilde\eta_u(\Delta)$; recall that

$$\widetilde J_{(k)}(\Delta,u)=\mathbf{M}\tilde\eta_u(\Delta)\,[\tilde\eta_u(\Delta)-1]\cdots[\tilde\eta_u(\Delta)-k+1].$$

Theorem 2. Under a coordinated increase $(u,|\Delta|)\to\infty$,

$$|\Delta|=\frac{2\pi\mu}{\sqrt{-\rho''(0)}}e^{u^2/2},$$

and assuming conditions (1), (2) are satisfied,

$$\lim_{(u,|\Delta|)\to\infty}\widetilde J_{(k)}(\Delta,u)=\mu^k,\qquad k=1,2,\dots. \tag{6}$$

In the proof of Theorem 2, assertion (5) is used, as well as the fact that the limiting behavior of $\widetilde J_{(k)}$ and $J_{(k)}$ is the same. Here the method used in the proof of Theorem 1 of the author' s paper ([1]) proves useful. The main result of the paper is obtained as a consequence of the theorem on convergence of moments (([2]), p. 198).

Theorem 3. If, for the correlation function $\rho(t)$ of a stationary Gaussian process, (1), (2) are satisfied, then under a coordinated increase $(u, |\Delta|) \to \infty$,

$$\lim_{(u,|\Delta|)\to\infty} \mathbf{P}\{\eta_u(\Delta) = k\} = \frac{(\mu|\Delta|)^k}{k!} e^{-\mu|\Delta|}, \qquad k = 0, 1, \dots .$$

It can be shown that the joint multidimensional distributions will also be Poisson. Theorem 3 is a generalization of Cramér' s result ([3]), in which the existence of four derivatives of the correlation function $\rho(t)$ and $\rho(t) = o(t^{-\varepsilon})$, $t \to \infty$, $\varepsilon > 0$, are assumed. Using Theorem 3, one can obtain a generalization of one more result of Cramér ([4]).

Theorem 4. If conditions (1), (2) are satisfied, then

$$\lim_{T\to\infty} \mathbf{P}\left\{\max_{0\le t\le T} \xi_t \le \sqrt{2\ln T} + \frac{z-A}{\sqrt{2\ln T}}\right\} = e^{-e^{-z}}, \qquad A = \ln \frac{2\pi}{\sqrt{-\rho''(0)}}.$$

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Note: Figure translations are in progress. See original paper for figures.