ON THE DESCRIPTION OF GENERALIZED EXTENSIONS WITH ONE-DIMENSIONAL IMAGINARY COMPONENT OF THE DIFFERENTIATION OPERATOR WITHOUT SPECTRUM

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Abstract

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E. R. TSEKANOVSKII

ON THE DESCRIPTION OF GENERALIZED EXTENSIONS WITH ONE-DIMENSIONAL IMAGINARY COMPONENT OF THE DIFFERENTIATION OPERATOR WITHOUT SPECTRUM

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I. Let T be a closed operator acting in a Hilbert space $\mathcal{G}$, for which

$$T_0 \subset T, \qquad T^* \subset T_0^*, \tag{1}$$

where T_0 is a symmetric operator with dense domain. Consider the Hilbert space $\mathcal{G}_+ = D_{T_0^*}$ with scalar product

$$(f, g)_+ = (T_0^* f, T_0^* g)_0 + (f, g)_0 \qquad (f, g \in \mathcal{G}_+). \tag{2}$$

Construct a triple of spaces $\mathcal{G}_+ \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_-$, where $\mathcal{G}_-$ is the space of generalized elements generating antilinear functionals on $\mathcal{G}_+$ (see ([2],[4])).

In ([4],[5]) it was shown that T and T^* can always be extended to $\mathcal{G}_+ = D_{T_0^*}$ in such a way that the resulting extensions $T_{\mathcal{G}_+}$ and $T^\times_{\mathcal{G}_+}$ are adjoint to one another in the generalized sense; moreover, $T_{\mathcal{G}_+}$ was called a generalized extension of the operator T.

M. S. Livšic is responsible for ([1]) the well-known theorem that every quasidissipative extension without spectrum of a simple symmetric operator with deficiency index $(1, 1)$ is unitarily equivalent to the differentiation operator T

$$Tf = \frac{1}{i}\frac{df}{dx} \qquad (f(0) = 0,\ 0 \le x \le \eta),$$

$$T^* f = \frac{1}{i}\frac{df}{dx} \qquad (f(\eta) = 0,\ 0 \le x \le \eta). \tag{3}$$

It is therefore natural that it is of interest to give a description of all generalized extensions of the differentiation operator (3).

In the present paper we give a description of the generalized extensions $T_{\mathcal{G}_+}$ of this operator that have a one-dimensional imaginary component $(T_{\mathcal{G}_+} - T^{\times}_{\mathcal{G}_+})/i$, which makes it possible to single out in a natural way an important class of such extensions.

II. Let $\mathcal{G}_0 = L_2[0,\eta]$, and let

$$T_0 f = \frac{1}{i}\frac{df}{dx} \qquad (f(0) = f(\eta) = 0).$$

It is known ([3]) that $T_0^* f = \frac{1}{i}\frac{df}{dx}$, and $D_{T_0^*}$ consists of absolutely continuous functions on $[0,\eta]$ whose derivatives belong to $L_2[0,\eta]$. Obviously, the operators T and T^* satisfy condition (1). It follows from (2) that $\mathcal{G}_+ = W_2^{(1)}[0,\eta] = D_{T_0^*}$, and the scalar product in $\mathcal{G}_+$ is given by the formula

$$(f,g)_+ = \int_0^\eta f'(x)\,\overline{g'(x)}\,dx + \int_0^\eta f(x)\,\overline{g(x)}\,dx.$$

Let us construct the space $\mathcal{G}_-$, so that $\mathcal{G}_+ \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_-$.

Consider the family of operators acting from $\mathcal{G}_+$ into $\mathcal{G}_-$:

$$T_{G_+}(\xi) f = \frac{1}{i}\frac{df}{dx} + i f(0)\,[e^{i\xi\eta}\delta(x-\eta) - \delta(x)], \tag{4}$$

$$\begin{pmatrix} -\infty < \xi < +\infty \\ 0 \le x \le \eta \end{pmatrix}$$

$$T^{\times}_{G_+}(\xi) f = \frac{1}{i}\frac{df}{dx} + i f(\eta) e^{-i\xi\eta}\,[e^{i\xi\eta}\delta(x-\eta) - \delta(x)]; \tag{5}$$

here $\delta(x-\eta)$, $\delta(x)$ are generalized elements (delta functions) generating the functionals

$$(\delta(x-\eta), f)_0 = \overline{f(\eta)}, \qquad (\delta(x), f)_0 = \overline{f(0)}.$$

It is not hard to see that for each fixed ξ the operators $T_{G_+}(\xi)$ and $T^{\times}_{G_+}(\xi)$ are extensions of T and T^*, respectively, and moreover

$$(T_{G_+}(\xi) f, g)_0 = (f, T^{\times}_{G_+}(\xi) g)_0 \qquad (f, g \in \mathcal{G}_+).$$

In addition, it follows from (4), (5) that

$$\frac{T_{G_+}(\xi) - T^{\times}_{G_+}(\xi)}{i} f = (f, [e^{i\xi\eta}\delta(x-\eta) - \delta(x)])_0 \, J\, [e^{i\xi\eta}\delta(x-\eta) - \delta(x)]$$

$$(J = -1). \tag{6}$$

Relation (6) shows that for every ξ the imaginary component of the family of operators $T_{G_+}(\xi)$ is one-dimensional.

We shall show that the family of operators (4) exhausts all generalized extensions of the operator T on $\mathcal{G}_+ = D_{T_0^*}$ that have a one-dimensional imaginary component.

III. In what follows we shall need one theorem, whose proof may be found in ([5]) and whose statement we give for convenience.

Theorem 1. Let the operator T satisfy conditions (1). In order that the operators T and T^* be extendable to the whole space

$$\mathcal{G}_+ = D_{T_0^*} = D_T + D_{T^*}$$

in such a way that the resulting extensions T_{G_+} and $T^{\times}_{G_+}$ are adjoint to each other in the generalized sense, it is necessary and sufficient that there exist linear operators $P(D_{T^*} \to \mathcal{G}_-)$ and $Q(D_T \to \mathcal{G}_-)$ possessing the following properties:

$$\begin{aligned}
&1) \quad (Pf_2, g_2)_0 = (f_2, T^* g_2)_0, \qquad (Qf_1, g_1)_0 = (f_1, Tg_1)_0 \\
&\qquad\quad (f_1, g_1 \in D_T,\ f_2, g_2 \in D_{T^*}); \\
&2) \quad (Pf_2, g_1)_0 = (f_2, Qg_1)_0 \qquad (g_1 \in D_T,\ f_2 \in D_{T^*}); \\
&3) \quad P\varphi = T\varphi, \qquad Q\varphi = T^*\varphi \qquad (\varphi \in D_{T_0} = D_T \cap D_{T^*}).
\end{aligned} \tag{7}$$

Moreover,

$$\begin{aligned}
T_{G_+} f &= Tf_1 + Pf_2, \\
T^{\times}_{G_+} f &= Qf_1 + T^* f_2
\end{aligned} \qquad (f = f_1 + f_2,\ f_1 \in D_{T^*},\ f \in \mathcal{G}_+). \tag{8}$$

Theorem 2. There do not exist two distinct generalized extensions T_{G_+} and T'_{G_+} of the operator

$$Tf = \frac{1}{i}\frac{df}{dx} \qquad (f(0) = 0,\ 0 \le x \le \eta)$$

on $\mathcal{G}_+ = D_{T_0^*}$, for which the real parts would be extensions of the self-adjoint operator

$$A_0\psi = \frac{1}{i}\frac{d\psi}{dx} \qquad (\psi(0) + \theta\psi(\eta) = 0,\ |\theta| = 1).$$

Proof. Let

$$T_{G_+} = A + iB, \qquad T'_{G_+} = A' + iB', \tag{9}$$

where $A = (T_{G_+} + T^{\times}_{G_+})/2$, $A' = (T'_{G_+} + T'^{\times}_{G_+})/2$, etc.

From theorem (1) it follows that

$$T_{\mathfrak{G}_+} f = Tf_1 + Pf_2, \qquad T'_{\mathfrak{G}_+} f = Tf_1 + P'f_2,$$

$$T^{\times}_{\mathfrak{G}_+} f = Qf_1 + T^* f_2, \qquad T'^{\times}_{\mathfrak{G}_+} f = Q'f_1 + T^* f_2. \tag{10}$$

From (9) and (10) it follows that

$$Af = \tfrac{1}{2}(Tf_1 + T^* f_2 + Qf_1 + Pf_2),$$

$$(f = f_1 + f_2,\ f_1 \in D_T,\ f_2 \in D_{T^*}), \tag{11}$$

$$A'f = \tfrac{1}{2}(Tf_1 + T^* f_2 + Q'f_1 + P'f_2).$$

In (5) it was shown that $T_0^* f = Tf_1 + T^* f_2$ $(f = f_1 + f_2,\ f_1 \in D_T,\ f_2 \in D_{T^*})$. Therefore, taking (11) into account, we obtain

$$Af = \tfrac{1}{2}(T_0^* f + Qf_1 + Pf_2), \qquad A'f = \tfrac{1}{2}(T_0^* f + Q'f_1 + P'f_2). \tag{12}$$

Suppose now that A and A' are extensions of the self-adjoint operator A_0, i.e. $Af_0 = A_0 f_0,\ A'f_0 = A_0 f_0$ $(f_0 \in D_{A_0})$. Since $T_0 \subset A_0 \subset T_0^*$, we have

$$A_0 f_0 = \tfrac{1}{2}(A_0 f_0 + Qf_1^0 + Pf_2^0),$$

$$A_0 f_0 = \tfrac{1}{2}(A_0 f_0 + Q'f_1^0 + P'f_2^0), \qquad (f_0 = f_1^0 + f_2^0,\ f_1^0 \in D_T,\ f_2^0 \in D_{T^*},\ f_0 \in D_{A_0}).$$

From the last relations it follows that

$$(Q - Q')f_1^0 = -(P - P')f_2^0. \tag{13}$$

From equalities (7) and (13) it follows that

$$((Q-Q')f_1^0, g_1)_0 = -((P-P')f_2^0, g_1)_0 = 0,$$

$$((P-P')f_2^0, g_2)_0 = -((Q-Q')f_1^0, g_2)_0 = 0.$$

Hence

$$((Q-Q')f_1^0, g_1)_0 = 0, \qquad ((Q-Q')f_1^0, g_2)_0 = 0,$$

therefore,

$$((Q-Q')f_1^0, g)_0 = 0 \qquad (g = g_1 + g_2,\ g_1 \in D_T,\ g_2 \in D_{T^*})$$

and, consequently, $(Q-Q')f_1^0 = 0$. Similarly, $(P-P')f_2^0 = 0$. From (8) and (12) it follows that $T_{\mathfrak{G}_+} f_0 = T'_{\mathfrak{G}_+} f_0,\ Af_0 = A'f_0$.

Thus, we have proved that the generalized extensions $T_{\mathfrak{G}_+}$ and $T'_{\mathfrak{G}_+}$ of the operator T coincide on the subspace D_{A_0} under the condition that their real parts are extensions of the self-adjoint operator A_0. We shall now show that $T_{\mathfrak{G}_+}$ and $T'_{\mathfrak{G}_+}$ coincide on the whole space $\mathfrak{G}_+$.

Let f_1^0 be an arbitrary function from D_T. Then, as is easy to see, one can choose a function $f_2^0 \in D_{T^*}$ such that $f_0 = f_1^0 + f_2^0$ belongs to the set D_{A_0}. From (9) it follows that $T^\times_{\mathfrak{G}_+} = A - iB,\ T'^\times_{\mathfrak{G}_+} = A' - iB'$. Therefore

$$Tf_1^0 = Af_1^0 + iBf_1^0, \qquad Tf_1^0 = A'f_1^0 + iB'f_1^0,$$

$$T^*f_2^0 = A'f_2^0 - iB'f_2^0, \qquad T^*f_2^0 = Af_2^0 - iBf_2^0. \tag{14}$$

From relations (14) it follows that

$$Tf_1^0 + T^*f_2^0 = A(f_1^0 + f_2^0) + iB(f_1^0 - f_2^0),$$

$$Tf_1^0 + T^*f_2^0 = A'(f_1^0 + f_2^0) + iB'(f_1^0 - f_2^0);$$

whence it follows immediately that

$$(A-A')f_0 + i(B-B')(f_1^0 - f_2^0) = 0 \qquad (f_0 = f_1^0 + f_2^0,\ f_0 \in D_{A_0}). \tag{15}$$

Since, as we have shown, $T_{\mathfrak{g}_+} f_0 = T'_{\mathfrak{g}_+} f_0$, it already follows easily that $Bf_0 = B'f_0$, $Af_0 = A'f_0$. Therefore from (15) $(B - B')(f_1^0 - f_2^0) = 0$, i.e.,

$$(B - B')f_1^0 = (B - B')f_2^0. \tag{16}$$

Since $f_1^0 + f_2^0 = f_0$, we have

$$(B - B')f_1^0 + (B - B')f_2^0 = 0. \tag{17}$$

From (16) and (17) we obtain that $(B - B')f_1^0 = 0$ for an arbitrary function $f_1^0 \in D_T$. Similarly one proves that $(B - B')f_2^0 = 0$ for an arbitrary function $f_2^0 \in D_{T^*}$. Hence it follows that $(B - B')f = 0$ $(f \in \mathfrak{G}_+)$. Since, when the imaginary components coincide, $B = B'$, the generalized extensions $T_{\mathfrak{g}_+}$ and $T'_{\mathfrak{g}_+}$, as is known (5), also coincide, it follows that $T_{\mathfrak{g}_+} = T'_{\mathfrak{g}_+}$. The theorem is proved.

Theorem 3. In order that a generalized extension $T_{\mathfrak{g}_+}$ of the operator

$$Tf = \frac{1}{i}\frac{df}{dx} \quad (f(0) = 0,\ 0 \le x \le \eta)$$

have a one-dimensional imaginary component, it is necessary and sufficient that the real part $(T_{\mathfrak{g}_+} + T^{\times}_{\mathfrak{g}_+})/2$ be an extension of some self-adjoint operator A_0 which is an extension of the operator

$$T_0 f = \frac{1}{i}\frac{df}{dx} \quad (f(0) = f(\eta) = 0,\ 0 \le x \le \eta).$$

Proof. It is known (3) that all self-adjoint extensions of the operator T_0 have the form

$$T(\theta)f = \frac{1}{i}\frac{df}{dx} \quad (f(0) + \theta f(\eta) = 0,\ |\theta| = 1,\ 0 \le x \le \eta). \tag{18}$$

Let now $T_{\mathfrak{g}_+}$ be some generalized extension of the operator T, for which $(T_{\mathfrak{g}_+} + T^{\times}_{\mathfrak{g}_+})/2$ is an extension of the operator $T(\theta)$ for fixed θ $(|\theta| = 1)$. From (4) it follows that ξ can be chosen so that $(T_{\mathfrak{g}_+}(\xi) + T^{\times}_{\mathfrak{g}_+}(\xi))/2$ will also be an extension of $T(\theta)$ for the same θ. By virtue of Theorem 2, $T_{\mathfrak{g}_+}$ and $T^{\times}_{\mathfrak{g}_+}(\xi)$ must coincide. From (6) it follows that $T_{\mathfrak{g}_+}$ has a one-dimensional imaginary component. Sufficiency is proved.

Let now $T_{\mathfrak{g}_+}$ be a generalized extension of the operator T having a one-dimensional imaginary component, i.e.,

$$T_{\mathfrak{g}_+} = A + i(\,\cdot\,, e_0)_0 J e_0 \quad (A = A^{\times},\ J = \pm 1). \tag{19}$$

From (19) it follows that

$$Tf_1 = Af_1 + i(f_1, e_0)_0 Je_0, \quad T^* f_2 = Af_2 - i(f_2, e_0)_0 Je_0, \quad (f_1 \in D_T,\ f_2 \in D_{T^*}),$$

hence

$$Af = T_0^* f - i(f_1 - f_2, e_0)_0 Je_0 \quad (f = f_1 + f_2,\ f_1 \in D_T,\ f_2 \in D_{T^*}). \tag{20}$$

It is not difficult to show that for any $f_1 \in D_T$ one can indicate such an $f_2 \in D_{T^*}$ that $(f_1, e_0)_0 = (f_2, e_0)_0$. Therefore the operator A, as is seen from (20), will act on vectors $f = f_1 + f_2$ $((f_1, e_0)_0 = (f_2, e_0)_0)$ as an operator in the space $\mathfrak{G}_0$ and will be a symmetric extension of T_0. Consequently, as is known (3), it must coincide with one of the operators $T(\theta)$ (see (18)). The theorem is proved.

Corollary. The totality of generalized extensions with one-dimensional imaginary component of the differentiation operator T (3) consists of the family (4).

Let us note in conclusion that all generalized extensions (4) in the space $\mathfrak{G}$ also have no spectrum in the finite part of the plane (see (5)).

Donetsk State University

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Note: Figure translations are in progress. See original paper for figures.