ON ESSENTIAL VARIABLES OF FUNCTIONS OF THE ALGEBRA OF LOGIC

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ON ESSENTIAL VARIABLES OF FUNCTIONS OF THE ALGEBRA OF LOGIC

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1°. In the present note we investigate the question of essential variables of functions of the algebra of logic (f.a.l.). In what follows, by f.a.l. we shall mean functions that depend essentially on all their variables (see (^{1})).

To make the formulation of the problem more precise, we introduce the following definitions.

Definition 1. A variable x_i $(1 \leq i \leq n)$ of a function $f(x_1, \dots, \dots, x_n)$ $(n > 1)$ is called a **strongly essential variable** (s.e.v.) if there exists a constant c such that, upon substituting this constant in place of x_i, one obtains a function depending essentially on $(n-1)$ variables.

Definition 2. A pair of variables (x_i, x_j) $(1 \leq i < j \leq n)$ of $f(x_1, \dots, x_n)$ $(n > 1)$ is called **separable** if there exists a set of $(n-2)$ constants such that, upon substituting these constants in place of all variables of $f(x_1, \dots, x_n)$ except x_i and x_j, one obtains a function depending essentially on x_i and x_j.

Definition 3. A variable x_i $(1 \leq i \leq n)$ of a function $f(x_1, \dots, \dots, x_n)$ $(n > 1)$ is called a **variable of order** k if there exist variables $x_{i_1}, \dots, x_{i_k}$ $(0 \leq i_1 < \dots < i_k \leq n,\ i_t \neq i\ (t = 1, 2, \dots, k))$ such that each pair (x_i, x_{i_t}) $(t = 1, 2, \dots, k)$ is separable and, whatever $j \neq i_t$ $(t = 1, 2, \dots, k)$, $j \neq i$, the pair (x_i, x_j) is not separable.

In (^{2}-^{4}) it was proved that every f.a.l. has at least one s.e.v. In (^{3}, ^{4}) it is proved that every variable of an f.a.l. is of at least first order.

We are interested in the problem of the number of s.e.v.' s of an arbitrary f.a.l. and of a lower estimate for the orders of the variables of an f.a.l.

2°. Let $L(f)$ be the number of s.e.v.' s of the function $f(x_1, \dots, x_n)$, and

$$L(n) = \min_{f(x_1, \dots, x_n)} L(f).$$

The function $x_1 \cdot x_2 \cdot \ldots \cdot x_{n-2} \cdot x_{n-1} \vee \bar{x}_1 \cdot \bar{x}_2 \cdot \ldots \cdot \bar{x}_{n-2} \cdot x_n$, obviously, has exactly two s.e.v.' s: x_{n-1} and x_n. Thus, $L(n) \leq 2$.

Let $f(x_1, \ldots, x_n)$ be an arbitrary f.a.l. Obviously, in order that $f(x_1, \ldots, x_n)$ depend essentially on all its variables, it is necessary and sufficient that the representation of $f(x_1, \ldots, x_n)$ in the form of a Zhegalkin polynomial contain an occurrence of each of the variables x_i $(1 \leq i \leq n)$ (see ([3])).

Next we note that the number of s.e.v.' s of $f(x_1, \ldots, x_n)$ does not change when some (possibly all) variables are replaced by their negations.

Lemma 1. If $f(x_1, \ldots, x_n)$ is an f.a.l. whose Zhegalkin polynomial contains a conjunction of rank n (1), then every variable of $f(x_1, \ldots, x_n)$ is an s.e.v.

Lemma 2. If there exists an $f(x_1, \ldots, x_n)$ having exactly one s.e.v., then there will be found an $f^*(x_1, \ldots, x_n)$ having exactly one s.e.v. and whose Zhegalkin polynomial contains a conjunction of rank n.

From Lemmas 1 and 2 we obtain that $L(n) \geq 2$. Thus, it is proved:

Theorem 1. $L(n) = 2$.

2°. Let B_k be the set of all sequences of length k of 0's and 1's. If $\tilde{\sigma} = (\sigma_1, \ldots, \sigma_k)$, then denote

$$|\tilde{\sigma}| = \sum_{i=0}^{k} \sigma_i \cdot 2^i.$$

Lemma 3. In order that the variable x_i $(1 \leq i \leq n)$ of the function $f(x_1, \ldots, x_n)$ be a variable of order k, it is necessary and sufficient that $f(x_1, \ldots, x_n)$ can be represented in the form

$$\begin{aligned} f(x_1, \ldots, x_n) = x_i \cdot & \bigvee_{\tilde{\sigma}=(\sigma_1, \ldots, \sigma_k) \subset A} x_{i_1}^{\sigma_1} \cdots x_{i_k}^{\sigma_k} \vee \\ & \vee \ \bar{x}_i \cdot \bigvee_{\tilde{\delta}=(\delta_1, \ldots, \delta_k) \subset B} x_{i_1}^{\delta_1} \cdots x_{i_k}^{\delta_k} \vee \\ & \vee_{\tilde{\gamma}=(\gamma_1, \ldots, \gamma_k) \subset C} \bigvee x_{i_1}^{\gamma_1} \cdots x_{i_k}^{\gamma_k} \cdot f_{|\tilde{\gamma}|}(x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i_k-1}, \\ & \qquad x_{i_k+1}, \ldots, x_n), \end{aligned} \tag{1}$$

where $A \cup B \cup C = B_k$, pairwise disjoint; $A \cup B$ is nonempty, and $f_{|\tilde{\gamma}|}$ are certain Boolean-algebra functions essentially depending on no more than $(n - k - 1)$ variables.

Let us outline the proof. If $f(x_1, \ldots, x_n)$ is representable in the form (1), then every pair (x_i, x_{i_t}) $(t = 1, 2, \ldots, k)$ is distinguished, and there are no other distinguished pairs of the form (x_i, x_j) $(j \neq i,\ j = i_t)$ $(t = 1, 2, \ldots, k)$.

Now suppose that $f(x_1, \dots, x_n)$ has a variable of k-th order. Without loss of generality this variable may be assumed to be x_1 (renaming the variables if necessary). Let $x_2, \dots, x_{k+1}$ be such variables of $f(x_1, \dots, x_n)$ that each pair (x_1, x_i) $(i = 2, \dots, k+1)$ is distinguished and there are no other distinguished pairs of the form (x_1, x_j) $(j > k+1)$; this can also be achieved by renaming the variables. There exist $\delta_2, \dots, \delta_n$ such that

$$f(0, \delta_2, \dots, \delta_n) \neq f(1, \delta_2, \dots, \delta_n).$$

Then, whatever $j > k+1$ may be,

$$f(0, \delta_2, \dots, \delta_{k+1}, \dots, \delta_j, \dots, \delta_n) = f(0, \delta_2, \dots, \delta_{k+1}, \dots, \bar{\delta}_j, \dots, \delta_n).$$

Consequently,

$$\begin{aligned} f(0, \delta_2, \dots, \delta_{k+1}, x_{k+2}, \dots, x_n) &= a, \\ f(1, \delta_2, \dots, \delta_{k+1}, x_{k+2}, \dots, x_n) &= \bar{a}. \end{aligned}$$

From this the assertion of the lemma is easily obtained.

Corollary 1. If a Boolean-algebra function $f(x_1, \dots, x_n)$ has a variable of order k, then there is at least one variable of order

$$p \geq \frac{(n-k-1)}{(2^k - 1)}.$$

Corollary 2. Every Boolean-algebra function of n variables $(n > 3)$ has no more than one variable of order 1.

Theorem 2. Whatever the Boolean-algebra function of n variables $(n > 2)$ may be, it has at least one variable of order

$$k \geq \left[\left(1 - \frac{1}{\log_2 \log_2 n} \right) \log_2 n \right].$$

Proof. Let k be the greatest of the orders of the variables of $f(x_1, \dots, x_n)$. Then, by Corollary 1, we have

$$\frac{n-k-1}{2^k - 1} < k.$$

Hence we obtain that

$$k \geq \left[\left(1 - \frac{1}{\log_2 \log_2 n} \right) \log_2 n \right].$$

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Note: Figure translations are in progress. See original paper for figures.

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