Differential equations $yy^{'} = Q_4(x, y)$, having an ellipse as limit cycle

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Date: 1967-01-01T00:00:00+00:00

Abstract

In this work, necessary and sufficient conditions are found for the existence of closed algebraic curves of the second order among the trajectories of the equation

$$yy' = Q_4(x, y), \tag{1}$$

where $Q_4(x, y)$ is a polynomial of the fourth degree. At the same time, it is shown that two distinct ellipses cannot (simultaneously) be limit cycles of the equation (1), and all cases where an ellipse is a rough limit cycle of this equation are specified.

Considering special cases, the author shows that there exist equations (1) which, along with an ellipse, have at least two more limit cycles, as well as multiple and singular cycles (in the form of a separatrix loop going from a saddle to the same saddle). Furthermore, in a number of cases, regions of cycle uniqueness are identified in the coefficient space. In §2, conditions are specified under which the equation (1) has a separatrix loop of the saddle $O(0, 0)$ such that within the simply connected region bounded by this loop, there are two singular points $A(-d, 0)$ and $B(d, 0)$ of the focus type, while outside this loop, the integral curves of the equation (1) are closed.

Bibliography: 10 items.

Full Text

Preamble

This work, published in 1967, investigates the qualitative behavior of solutions to differential equations of the form $yy' = Q(x, y)$, building upon the foundational methods established in [6] and [9]. We consider the system:

$$y^2 = ax^2 + 2bx + c \quad (a < 0) \tag{1}$$

Following the analysis in [6], we define the auxiliary functions and coefficients $a_{03}, a_{04}, a_{13}, a_{12}$, and parameters $\alpha, \beta, \gamma, \delta, \mu$. The system can be transformed into the following form:

$$y' = \frac{(P + \alpha y + \gamma x + \delta xy + \sigma x^2 + \mu y^2)(c + 2bx + ax^2 - y^2) + ax + b}{Q(x, y)} \tag{2}$$

where the coefficients are determined by the boundary conditions and the geometry of the phase plane.

§ 1. Qualitative Analysis of the System

We examine the equilibrium points and the existence of limit cycles for the system described by equation (1). Let us consider the case where $\mu = 0$. Under these conditions, the system (1) can be simplified to:

$$y' = b + ax - \mu(c + 2bx + ax^2 - y^2) \tag{3}$$

As demonstrated in [6], the existence of a limit cycle L is contingent upon the sign of the discriminant and the nature of the singular points. Specifically, if $h \neq 0$, where h is defined by the relation:

$$h = \frac{\xi_0 (b^2 - ac)^2 \cos(\theta - \alpha)^2}{(ac - b^2)\sqrt{(ac - b^2)(b^2 - a)}} \tag{4}$$

then the system exhibits distinct topological structures in the domain D.

Lemma 1. If $b = 0$ and $a \neq 0$, the limit cycle L is unique and stable, as shown in [7].

Lemma 2. In the domain D, if $b \neq 0$, then for $a = 0$, the system (4) possesses no limit cycles. The symmetry of the function $f(x, y) = f(x, -y)$ implies that the integral curves are symmetric with respect to the x-axis. By calculating the difference $\Delta(x, y) = Q(x, y) - Q(x, -y) = 2y(a + bx)(c + 2bx + ax^2 - y^2)$, we can determine the stability of the equilibrium points.

§ 2. Limit Cycles and Bifurcations

We further investigate the system (4) under the condition $y > 1$. Let the singular points be $O(0, 0)$, $A(-d, 0)$, and $B(d, 0)$. The behavior of the trajectories near $O(0, 0)$ is governed by the eigenvalues of the linearized system. For $a = 0$, the points A and B merge or change their stability characteristics.

When $|a| < 1$, the system exhibits a nested structure of trajectories. We define the Dulac function $N(x, y)$ to prove the non-existence of limit cycles in certain subdomains. Using the criterion $M'_x + N'_y \neq 0$, we establish that:

$$M'_x(d,0) + N'_y(d,0) = -L(Q_x - Q_y + \dots) \tag{13}$$

For $a = -d$, the point B becomes a complex singular point. As a varies, the system undergoes a Hopf bifurcation, leading to the creation or destruction of limit cycles.

§ 3. Special Cases and Numerical Results

Consider the specific case where $b = \beta = c = \mu = 0$ and $a = -1$. The system reduces to:

$$y' = -x + (ay + \gamma x + xy + y^2)(1 - x^2 - y^2) \tag{15}$$

In this configuration, the system possesses a singular point at the origin $O(0,0)$. The stability of this point is determined by the sign of a. If $a > 0$, the origin is an unstable focus; if $a < 0$, it is a stable focus. The transition at $a = 0$ corresponds to the emergence of a limit cycle L_2.

Our analysis shows that for $y < 0$, the trajectories are attracted to the singular points A and B. The global phase portrait is constructed by stitching together the local behaviors in the regions $y > 1$, $0 < y < 1$, and $y < 0$.

Theorem 3. For the system (9), if the parameters belong to the set $G_1 \cup G_2$, there exists at least one limit cycle surrounding the origin. If the parameters transition into region G_3, the limit cycle disappears through a homoclinic bifurcation involving the points A and B.

References

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Note: Figure translations are in progress. See original paper for figures.