AN ANALOGUE OF THE CARTAN-WEYL THEORY FOR INFINITE-DIMENSIONAL REPRESENTATIONS OF A SEMISIMPLE COMPLEX LIE GROUP

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AN ANALOGUE OF THE CARTAN–WEYL THEORY FOR INFINITE-DIMENSIONAL REPRESENTATIONS OF A SEMISIMPLE COMPLEX LIE GROUP

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In the classical Cartan–Weyl theory, a well-known description is given of finite-dimensional irreducible representations of a semisimple complex connected Lie group; it may be stated as follows. Let a group G with Cartan subgroup D and triangular root subgroups Z_-, Z_+ be under consideration. Let $g \mapsto T_g$ be an irreducible finite-dimensional representation of the group G in a space E. Then there exists a unique vector $e_0 \in E$, $e_0 \neq 0$, determined up to normalization and fixed with respect to the group Z_+:

$$T_z e_0 = e_0, \qquad z \in Z_+.$$

In addition, the vector e_0 is an eigenvector with respect to the diagonal subgroup D,

$$T_\delta e_0 = a(\delta) e_0, \qquad \delta \in D,$$

and the eigenvalue $a(\delta)$ (a character of the group D) determines the representation T_g up to equivalence. The character $a(\delta)$ is called the **highest weight**, and the vector e_0 the **highest vector** of the representation T_g. It is known ([1,2]) how, with the aid of the highest vector, to obtain a convenient realization of the representation T_g. An analogous construction may be carried out for the group Z_- (the lowest weight and the lowest vector of the representation T_g).

In the present note an analogous result will be obtained for infinite-dimensional representations of the group G. For such representations we shall have to introduce a somewhat new concept of extremal irreducibility. It turns out that every extremally irreducible representation of the group G possesses a generalized highest vector, situated in the dual space E'. However, a curious feature of infinite-dimensional representations is the fact that the highest vector is now no longer determined uniquely. The number of highest vectors is finite and is connected with certain symmetry properties determined by the Weyl group. Moreover, for extremally irreducible representations one obtains a complete classification up to present (strong) equivalence.

On the other hand, in the theory of nonunitary representations an important role is played by the concept of "complete irreducibility," originally introduced by R. Godement ([1]). In the work ([3]), M. A. Naimark and the author gave a description of all completely irreducible representations of the group G up to a certain "weak" equivalence. We can now give a further refinement of this result in terms of strong equivalence. To each highest weight $a(\delta)$ there is associated a certain bundle of linear spaces, obtained from one space by varying the topology, and every completely irreducible representation of the group G with highest weight $a(\delta)$ acts in one of the spaces of this bundle.*

Let us pass to a more detailed exposition of the results.

I. Basic definitions

By the symbol $\pi = (T_g, E, \tau)$ we shall denote a continuous representation T_g in a locally convex space E, complete or at least quasicomplete, with topology τ.

Definition 1. We shall call a representation π^0 a **subordinate** representation of π if the space E^0 is contained in E as an invariant

* All these results once again emphasize the naturalness of studying nonunitary representations in the class of general linear topological spaces.

a subspace $\neq E$ and the topology τ^0 is no weaker than the topology τ. If, moreover, E^0 is everywhere dense in E, then we shall say that π^0 is a **kernel** of the representation π, and π is its envelope.

Definition 2. We shall call a representation ε **extremally irreducible** if: 1) every representation subordinated to ε coincides with ε; 2) every weakly continuous operator commuting with all operators of ε is a multiple of the identity.

Definition 3. Two representations π^1, π^2 are called (strongly) **equivalent** if there exists a linear homeomorphism intertwining the operators of these representations ($AT_g^1 = T_g^2 A$). The representations π^1, π^2 are called **weakly equivalent** if they have equivalent kernels.

It may be agreed that equivalent kernels are identified, and then any two weakly equivalent representations may be regarded as different envelopes of one and the same kernel. If one restricts oneself to the class of countably normed spaces, then from Banach' s theorem it is easy to obtain the following assertions: the topology τ in which a representation is extremally irreducible is determined uniquely; in the definition of extremal irreducibility, property 2) follows from 1) if the operator in question has at least one point of the spectrum.

It is of interest to note one further circumstance: *if a representation of the group G is extremally irreducible, then it is infinitely differentiable* (indeed, every continuous representation of a Lie group contains a differentiable kernel, which in our case must coincide with the entire representation).

It will be convenient for us to give the definition of a generalized highest vector and of the infinitesimal form:

Definition 4. Let π be a differentiable representation of the group G in a space E. A linear functional $f \in E'$ will be called a (generalized) **highest vector** of the representation π if f satisfies the system of equations:

$$f(\mathcal{D}\_x) = 0, \qquad f(\mathcal{D}_0 x) = \lambda f(x), \qquad \lambda = \lambda(\mathcal{D}_0),$$

where $\mathcal{D}\_$, $\mathcal{D}_0$ are the infinitesimal operators in E generated respectively by the subgroups $Z\_$, D.

Considering the eigenvalue λ as a linear form on the Lie algebra of the group D, we shall agree to say that λ is the **weight** of the vector f and the **highest weight** of the representation π. In what follows, in describing representations of the group G, we shall need only such weights $\lambda = p^i_{it} + q_i \bar{t}^i$ that the difference $p-q$, where $p = \{p_i\}$, $q = \{q_i\}$ and t^i are canonical coordinates in the Lie algebra of the group D, belongs to the discrete weight lattice of the maximal compact subgroup $\mathfrak{u} \subset G$. We shall agree to call every pair of vectors $\alpha = (p, q)$ possessing this property a **signature**. The signatures are in one-to-one correspondence with the characters $\alpha(\delta)$ of the Cartan subgroup D.

II. **Minimal representations of the group G.** Let α be an arbitrary signature and $e(\alpha)$ the elementary representation of the group G ([4]), realized in the space $\mathfrak{D}_\alpha$, consisting of infinitely differentiable functions on the group $\mathfrak{u}$*. According to the results of ([4]), the representation $e(\alpha)$ is always irreducible, except for those cases when at least one pair of numbers

$$p_\omega = 2(p, \omega)/(\omega, \omega), \qquad q_\omega = 2(q, \omega)/(\omega, \omega),$$

where ω is a root in the Lie algebra of the group G, consists of nonzero integers of the same sign; in this case the signature α is called **singular**. If the case of singularity occurs, then the representation $e(\alpha)$

* The representation $e(\alpha)$ is induced by the character $\alpha(\delta) = \alpha(k)$ of the maximal decomposable subgroup $K \subset G$ in the class of infinitely differentiable functions on G/K.

is completely reducible and contains only a finite number of irreducible components. It turns out that, in order to enumerate these components (up to equivalence), it suffices to consider only those which, upon restriction to the subgroup U, contain the minimal among the highest weights (relative to this subgroup) realized in the space $\mathfrak{D}_\alpha$. Such components are completely determined by specifying the signature α. We denote them by $\mu(\alpha)$ and call them **minimal representations** of the group G. Two minimal representations $\mu(\alpha)$,

$\mu(\beta)$ are equivalent if and only if there exists a transformation s from the Weyl group $\mathfrak{S}$ such that $\beta = s\alpha$, i.e. $\beta = (sp, sq)$ for $\alpha = (p, q)$. In this case it is always possible to choose a signature α_0, lying on the orbit $\mathfrak{S}_\alpha$, such that the representation $\mu(\alpha_0)$ is realized in the invariant subspace $\mathfrak{R}_{\alpha_0} \subset \mathfrak{D}_{\alpha_0}$. We note that the following holds.

Proposition 1. *The minimal representation $\mu(\alpha_0)$ possesses a generalized highest vector of weight α_0.*

III. Scheme for the classification of extremally irreducible representations. The basis of our classification is the following

Proposition 2. *Every extremally irreducible representation of the group G is infinitesimally equivalent to one of the minimal representations $\mu(\alpha)$.*

Proof of this proposition is analogous to the work ([3]); however, instead of the group algebra X we now consider the universal enveloping algebra U of the group G and study the structure of its maximal ideals. From this we obtain, as a consequence:

Proposition 3. *Every extremally irreducible representation of the group G possesses a generalized highest vector.*

Proof is based on the observation that the generalized highest vector can be defined by means of an infinite system of linear equations (written in the canonical basis), and such a system is completely determined by the infinitesimal structure of the representation.

Proposition 4. *If an irreducible differentiable representation π possesses a generalized highest vector of weight α, then it can be realized as the kernel of the representation $\mu(\alpha)$.*

Proof reduces to the standard method of embedding irreducible representations into the regular representation of the group G.

Finally, one verifies

Proposition 5. *The representation $\mu(\alpha)$ is extremally irreducible.*

Proof is based on considering the group algebra X of all finite infinitely differentiable functions on the group G. It turns out that the representation $\mu(\alpha)$, transferred in the usual way to the algebra X, is algebraically irreducible (here a theorem of the Pelley–Wiener type for the algebra X is used; see ([5])), and our assertion follows from this.

We can now formulate the final result:

Theorem 1. *Let ε be an extremally irreducible representation of a semisimple complex connected group G in a Fréchet-type space. Then there exists a signature α such that ε is equivalent to $\mu(\alpha)$. Moreover, the representation ε possesses a generalized highest vector of weight α.*

Proof follows immediately from Propositions 1–5; moreover, we use the well-known fact of the uniqueness of the topology in a Fréchet space.

The conditions of belonging to the Fréchet type can easily be discarded, and then one obtains

Theorem $1'$**.** *Every extremally irreducible representation of the group G in a locally convex vector space E can be realized in one of the spaces $\mathfrak{R}_\alpha$ by the formula of the representation $\mu(\alpha)$.*

However, we do not know whether every locally convex topology in $\mathfrak{R}_a$, with respect to which $\mathfrak{R}_a$ is complete and $\mu(a)$ is continuous, coincides with the topology of the class of infinitely differentiable functions in $\mathfrak{R}_a$.

IV. **The question of uniqueness of the highest vector. Another formulation of the main result.** It is not difficult to prove that the following holds.

Proposition 6. *The given extremally irreducible representation may have several different highest weights. They all have the form $\alpha_s = (sp, sq)$, where s is some transformation of the Weyl group. Accordingly, the number of linearly independent highest vectors varies from 1 to N, where N is the order of the Weyl group.*[1]

We now define the infinitesimal character χ of an extremally irreducible representation of the group G as the eigenvalue $\chi(\mathfrak{z})$ of the operators of the center $\mathfrak{Z}$ of the universal enveloping algebra of the group G:

$$A_{\mathfrak{z}} = \chi(\mathfrak{z}) \cdot I,$$

where $A_{\mathfrak{z}}$ is the analytic operator corresponding to the element $\mathfrak{z} \in \mathfrak{Z}$. Then the following holds.

Proposition 7. *Every extremally irreducible representation of the group G in a Fréchet space is determined uniquely up to equivalence by its infinitesimal character $\chi(\mathfrak{z})$ and by the minimal one of the highest weights k_0 entering into the decomposition of the given representation into irreducible representations of the maximal compact subgroup $\mathfrak{U}$.*

It is also not difficult to establish a correspondence between extremally irreducible representations and orbits in the Lie algebra (more precisely, in its complex envelope) with respect to the adjoint representation of the group G.

V. **Description of all completely irreducible representations.** Let us introduce in the space $\mathfrak{R}_a$ the strongest topology with respect to which the operators $\mu(a)$ still form a continuous representation, and put $\mathfrak{M}_\alpha = \mathfrak{R}'_{\alpha'}$, where

[1] This number depends on the degree of degeneration of the signature $\alpha = (p, q)$. We shall not dwell here on a more precise estimate of it.

$\alpha' = (-\bar{q}, -\bar{p})$ and $\mathfrak{R}'$ denotes the linear space conjugate to $\mathfrak{R}$. Then there is an inclusion

$$\mathfrak{R}_a \subset H_a \subset \mathfrak{M}_a,$$

where H_a is the completion of $\mathfrak{R}_a$ in the topology of the Hilbert space $L^2(\mathfrak{U})$. Thus we have the standard model of a rigged Hilbert space. We shall call such a model an α-envelope (for fixed α) and say that a representation of the group G is contained in the given α-envelope if it can be realized in some space lying between $\mathfrak{R}_a$ and $\mathfrak{M}_a$, according to the formula $\mu(a)$. Now we can formulate a theorem which refines the result of work ([3]):

Theorem 2. *Every completely irreducible representation of the group G is contained in one of the α-envelopes.*

Thus the question of the classification of all irreducible representations of the group G is completely resolved (within the framework of the adopted definitions), and we obtain a remarkable analogy with the classical Cartan–Weyl theory.

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Note: Figure translations are in progress. See original paper for figures.