Some remarks concerning a differential equation of arbitrary order

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Abstract

For a certain class of autonomous differential equations of the n-th order, a three-parameter family of solutions is constructed, the derivative of which vanishes at infinity in the form of a Dirichlet series that converges uniformly and absolutely on the real semiaxis, starting from a certain negative number that is sufficiently small in absolute value. Bibliography: 2.

Full Text

Preamble

In 1967, in the journal *TM III* (Vol. 11, No. 517.911), and as previously discussed in [**?**] and [**?**], D. D. Baĭnov investigated the n-th order differential equation:

$$y^{(n)} + p_0 y^{(n-1)} + \sum_{l=0}^{n-4} \sum_{s=1}^{n-3-l} p_{ls} y^{(n-2-l-s)} y^{(s)} = 0, \quad (n > 6) \tag{1}$$

where p_0 and p_{ls} are constants, and $p_0 \neq 0$. We seek a solution to equation (1) in the form of a Dirichlet series:

$$y = \sum_{k=1}^{\infty} a_k e^{-kbx} \tag{2}$$

where a_k are coefficients to be determined, b is a constant such that $Re(b) > 0$, and the series converges for sufficiently large x.

By substituting the series (2) into equation (1), we obtain the following relation for the coefficients:

$$\sum_{k=1}^{\infty} a_k(-kb)^n e^{-kbx} + p_0 \sum_{k=1}^{\infty} a_k(-kb)^{n-1} e^{-kbx} + \sum_{l=0}^{n-4} \sum_{s=1}^{n-3-l} p_{ls} \left(\sum_{r=1}^{\infty} a_r (-rb)^{n-2-l-s} e^{-rbx} \right) \left(\sum_{m=1}^{\infty} a_m (-mb)^s e^{-mbx} \right)$$

Equating the coefficients of e^{-kbx} to zero, we derive the recurrence relations for a_k. For $k = 1$, we have:

$$a_1(-b)^n + p_0 a_1(-b)^{n-1} = 0$$

Since we assume $a_1 \neq 0$, it follows that $b = p_0$. For $k > 1$, the coefficients a_k are determined by:

$$a_k[(-kb)^n + p_0(-kb)^{n-1}] + \sum_{l=0}^{n-4} \sum_{s=1}^{n-3-l} p_{ls} \sum_{r=1}^{k-1} a_r a_{k-r} (-rb)^{n-2-l-s}(-(k-r)b)^s = 0 \tag{3}$$

From (3), we can express a_k as:

$$a_k = \frac{1}{k^{n-1}(k-1)p_0^n} \sum_{l=0}^{n-4} \sum_{s=1}^{n-3-l} p_{ls} \sum_{r=1}^{k-1} a_r a_{k-r} r^{n-2-l-s}(k-r)^s p_0^{n-2-l} \tag{4}$$

where $k = 2, 3, \ldots.$

To ensure the convergence of the series (2), we analyze the growth of the coefficients a_k. Let $|a_1| = A$. We aim to show that $|a_k| \leq A^k$ for all k. Using the properties of the beta function and the maximum values of the terms in the summation, we consider the integral:

$$\int_0^1 x^{n-2-l-s}(1-x)^s dx = \frac{(n-2-l-s)!s!}{(n-1-l)!}$$

By applying estimates to the summation in (4), we can establish bounds on $|a_k|$. Specifically, we utilize the fact that for $x \in [0, 1]$, the function $f(x) = x^k(1-x)^m$ reaches its maximum at $x = \frac{k}{k+m}$.

Following the methodology of D. D. Baĭnov, we evaluate the sum:

$$\sum_{l=0}^{n-4} \sum_{s=1}^{n-3-l} \frac{(n-2-l-s)!s!}{(n-1)!} \left(\frac{n-2-l-s}{n-2-l}\right)^{n-2-l-s} \left(\frac{s}{n-2-l}\right)^s$$

After simplification and applying the condition $n > 6$, it can be shown that if the coefficients of the original equation satisfy the condition:

$$\max(|p_0|, |p_{ls}|) < \frac{(n-1)(n-2)(n-3)}{4(n-2)^2 + (n-3)^2}$$

then the coefficients $|a_k|$ remain bounded such that the Dirichlet series (2) converges for $Re(p_0 x) > \ln A$.

Thus, we have demonstrated that under the specified constraints on the parameters p_0 and p_{ls}, the nonlinear differential equation (1) admits a solution in the form of a convergent Dirichlet series. This result extends the findings presented in [**?**] and [**?**] regarding the existence of analytic solutions for high-order nonlinear equations.

References

1. Baĭnov, D. D. "On a nonlinear differential equation of n-th order," *Journal of Mathematical Analysis*, No. 3, pp. 327-330, 1965.
2. Baĭnov, D. D. "Convergence of Dirichlet series solutions," *Reports of the Bulgarian Academy of Sciences*, Vol. 2, No. 6, pp. 853-854, 1966.

Note: Figure translations are in progress. See original paper for figures.