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MATHEMATICS

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Abstract

Full Text

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MATHEMATICS

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AN EXAMPLE OF AN OPEN COMPACT MONOTONE CONTINUOUS MAPPING OF A NONMETRIZABLE BICOMPACTUM ONTO A COMPACTUM

(Presented by Academician P. S. Aleksandrov, 27 II 1967)

V. V. Proizvolov posed the following problem. Let $f : X \rightarrow Y$ be an open compact mapping, X a bicom pactum, Y a compactum. Must the space X be metrizable? (see ⁽¹⁾).

We shall call a mapping $f : X \rightarrow Y$ **compact** if the preimage $f^{-1}y$ is compact (= a metrizable compact set) for every point $y \in Y$. A mapping f is called **monotone** if all $f^{-1}y$ are connected. We construct an example that gives a negative solution of the problem.

In the numerical plane R^2 consider the square $I = E\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Denote by P the set $E\{(x, y) : y \leq 0\}$. Let $S = I \setminus P$, $T = E\{(x, y) : y = -1\}$, $G = E\{(x, y) : y = 0\}$. On the set I consider the following topology t . At the points of S the topology induced on S by the topology of the numerical plane R^2 is preserved.

Let $p = (x, y) \in P \setminus (T \cup G)$ be an arbitrary point, $p_0 = (x, 0)$. In S construct two rays l_p and \bar{l}_p , issuing from the point p_0 , symmetric with respect to the line l_{p_0} passing through the point p_0 parallel to the axis OY , and such that the angle between l_p and l_{p_0} is equal to $|y|d$, where d is a right angle. Let q be an arbitrary point on l_p , and let \bar{q} be the point symmetric to it with respect to l_{p_0} (if $\bar{q} \notin S$, then for \bar{q} we take the point of intersection of the line \bar{l}_p with the side of the square). Denote by m and \bar{m} the lines in S perpendicular to l_p and \bar{l}_p at the points q and \bar{q} .

Let ε be a positive number, $\varepsilon \leq \min |y|, (1 - |y|)$. Introduce the notation: $E_{\varepsilon p} = E\{(x', y') : x' = x, |y - y'| < \varepsilon\}$, $p_1 = (x, y + \varepsilon)$, $p_2 = (x, y - \varepsilon)$. Further, let q_{p_1} and q_{p_2} , \bar{q}_{p_1} and \bar{q}_{p_2} be the points of intersection of the lines l_{p_1}, l_{p_2} and $\bar{l}_{p_1}, \bar{l}_{p_2}$ (which are constructed for the points p_1 and p_2 in the same way as the lines l_p and \bar{l}_p for the point p) with the lines m and \bar{m} . Denote by

Q_{mp} the set consisting of the interior points of the two triangles (p_0, q_{p_1}, q_{p_2}) and $(p_0, \bar{q}_{p_1}, \bar{q}_{p_2})$.

Put $O_{\varepsilon mp} = E_{\varepsilon p} \cup Q_{mp}$. By definition, the basis of the system of neighborhoods of the point p in the new space $\nabla = (I, t)$ will consist of sets of the type $O_{\varepsilon mp}$ for all possible $\varepsilon \leq \min |y|, (1 - |y|)$ and for arbitrarily chosen points q on l_p , determining the lines m .

If the point $p = p_0 \in G$, then $l_p = \bar{l}_p = l_{p_0}$, m is a line in S perpendicular to l_{p_0} at some point q , $\varepsilon \leq 1$, $p_1 = p_0$, Q_{mp_0} consists of the interior points of one triangle $(p_0, q_{p_2}, \bar{q}_{p_2})$, $E_{\varepsilon p_0} = E\{(x', y') : x = x', |y'| < \varepsilon, y' \leq 0\}$. Sets of the type $O_{\varepsilon mp_0} = E_{\varepsilon p_0} \cup Q_{mp_0}$ form, by definition, the basis of the system of neighborhoods of the point p_0 .

Finally, let the point $d = (x, -1) \in T$. Here $\varepsilon \leq 1$, $p_2 = d$, and l_d and \bar{l}_d lie on the segment G ; m and \bar{m} are lines parallel to the axis OY , meeting G at the points $q = (z, 0)$ and $\bar{q} = (\bar{z}, 0)$. The set Q_{md} consists of the interior points of the two triangles (p_0, q, q_{p_1}) and $(p_0, \bar{q}, \bar{q}_{p_1})$, $E_{\varepsilon d} = E\{(x', y') : x' = x, 1 - |y'| < \varepsilon, y' \geq -1\}$. Sets of the type $O_{\varepsilon md} = E_{\varepsilon d} \cup Q_{md}$ do not define here a basis of the system of neighborhoods.

Put $V_{md} = I\{(x', y') : (x', y') \in P, z < x' < \bar{z}, x' \neq x\}$. And only sets of the type $W_{\varepsilon md} = O_{\varepsilon md} \cup V_{md}$ will constitute the basis of the system of neighborhoods of the point $d \in T$.

Thus, a new topological space ∇ has been constructed (the correctness of its definition is easy to verify). We shall prove that ∇ is a nonmetrizable bicompactum. It is easy to establish that the space ∇ is Hausdorff.

Let M be an arbitrary infinite set lying in P , and let $p = (x, y)$ be a point of complete accumulation of M in the bicompactum I . If p is not a point of complete accumulation of M in the space ∇ , then the point $d = (x, -1)$ will be a point of complete accumulation of M in ∇ , as follows easily from the construction of the topology at the point d . Hence it follows that the space $P \subset \nabla$ is bicompact.

(A). Let the point $z = (x, 0) \in G$, and

$$L = E \mid (x', y') : x' = x, -1 \leq y \leq 0 \mid .$$

If, for each point $p \in L$, one arbitrarily chooses a basic neighborhood O_{emp} , then the set

$$V_z = \bigcup_{p \in L} (O_{emp} \cap K)$$

will be a neighborhood of the point z in the rectangle $K = S \cup G$ (in its natural topology). Indeed, projecting from the point z each O_{emp} onto the closed set

$$C = E \mid (x', y') : y' = 1 \mid \cup E \mid (x', y') : x' = -1, 1, y \geq 0 \mid ,$$

we obtain an open cover of the bicompactum C . From this cover choose a finite subcover, to each element of which there corresponds some O_{emp_i} , $i = 1, 2, \dots, n$,

projecting into it. Let j be such that the segment $l_{p_j} \cap O_{em}p_j$ has the smallest length d (among the segments $l_{p_i} \cap O_{em}p_i$, $i = 1, 2, \dots, n$). Then the open disk (in K) with center at the point z and radius d will be contained in $V_z \subset K$. It follows from what has been proved that, if the open (in K) sets V_z are chosen for each point $z \in G$, then the set

$$\bigcup_{z \in G} V_z$$

will be a neighborhood of the segment G in the bicom pactum K .

Now let $\Sigma = \{H\}$ be an arbitrary open cover of the space ∇ . There exists a finite number of sets $H_i \in \Sigma$, $i = 1, 2, \dots, n$, covering P :

$$P \subseteq \bigcup_{i=1}^n H_i.$$

From the assertion proved in (A) it follows that the set

$$A = \left(\bigcup_{i=1}^n H_i \right) \cap K$$

will be an open neighborhood of the set G in K ; therefore the set $B = K \setminus A$ is closed and hence bicom pact in K , and since it lies in S , it is bicom pact in ∇ . There is a finite number of sets H_j , $j = 1, 2, \dots, m$, such that

$$B \subseteq \bigcup_{j=1}^m H_j.$$

But then

$$\nabla = \bigcup_{i=1}^n H_i \cup \bigcup_{j=1}^m H_j.$$

It is proved that ∇ is a bicom pact space.

The space ∇ is not metrizable. Its subspace $S \cup G$ is completely regular, but not normal. This can be proved by the same method by which the nonnormality of the space of V. V. Nemytzkiï is proved (see, for example, (2)).

“Project” the bicom pactum ∇ onto the upper side

$$\Delta = E \mid (x, y) : y = 1 \mid$$

of the square I , i.e., assign to the point $(x, y) \in \nabla$ the point $(x, 1)$. This projection φ will be a continuous mapping of ∇ onto the compactum Δ . The preimage of the point $(x, 1)$ under this mapping will be the set

$$E \mid (x', y) : x' = x \mid,$$

homeomorphic to the segment $[0, 1]$. Consequently, φ is a compact monotone mapping. It is easy to verify that it is open. The example is constructed.

Let us call a bicom pactum that is mapped openly and compactly onto a compactum a **bicom pactum of type S** . The following are easily proved:

Theorem 1. *A countable product of bicom pacts of type S is a bicom pactum of type S .*

Theorem 2. *A topological (discrete) sum of bicom pacts of type S is a space that is perfectly openly and compactly mapped onto a metrizable space.*

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Note: Figure translations are in progress. See original paper for figures.

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