DIFFERENCE INEQUALITIES FOR FUNCTIONS HARMONIC IN A DISK

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Fig. 1

Figure 1: Fig. 1

Abstract

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MATHEMATICS

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DIFFERENCE INEQUALITIES FOR FUNCTIONS HARMONIC IN A DISK

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For definiteness, here the unit disk $Q(0 \le r \le 1,\ 0 \le d < 2\pi)$ is considered. The well-known theorem of Schwarz ([1]) states that, among all functions $U(r,\alpha)$ harmonic inside Q, bounded there in absolute value by one and vanishing at zero at the center of the disk, the function whose boundary values are $U(1,\alpha) = \varphi_0(\alpha - t)$ assumes the greatest value in absolute value at a given point $(r,t) \in Q$, where

$$\varphi_0(\alpha) = \frac{4}{\pi}\sum_{\nu=0}^{\infty}(-1)^\nu \frac{\cos(2\nu+1)\alpha}{2\nu+1} = \begin{cases} 1, & -\pi/2 < \alpha < \pi/2, \\ -1, & \pi/2 < \alpha < {}^3/2\,\pi. \end{cases}$$

We shall use the notation $U_f(r,\alpha)$ for the solution of the Dirichlet problem corresponding to the function $f(t)$ prescribed on the boundary of the disk Q. Then this assertion means that, under the conditions

$$|f(t)| \le 1, \tag{1}$$

$$\int_0^{2\pi} f(t)\,dt = 0 \tag{2}$$

the inequality

$$|U_f(r,t) - U_f(0,t)| \le |U_{\varphi_0}(r,0) - U_{\varphi_0}(0,0)| \tag{3}$$

is valid.

Fig. 1

If one considers the difference of the values of $U_f(r,t)$ at two arbitrary points (r_1,t) and (r_2,t) on some radius, then the following proposition holds, generalizing inequality (3):

Theorem 1. Let $0 \le r_1, r_2 < 1$. In order that, for $r_1 \ne r_2$, the inequality

$$|U_f(r_1,t) - U_f(r_2,t)| \le |U_{\varphi_0}(r_1,0) - U_{\varphi_0}(r_2,0)|$$

hold for every function $f(\alpha)$ satisfying conditions (1) and (2), it is necessary and sufficient that the point with coordinates (r_1,r_2) in the plane $r_1 r_2$ lie on the curve belonging to the square $0 < r_1, r_2 < 1$ with equation

$$r_1^2 r_2^2 - r_2^2 r_1 - r_1^2 r_2 - 2r_1 r_2 - r_2 - r_1 + 1 = 0$$

or below (to the left of) this part of the curve (in the shaded region in Fig. 1).

Let p be an arbitrary natural number; $W^{(p)}$ is the class of functions $f(t)$ of period 2π having an absolutely continuous derivative of order $p-1$ and such that $|f^{(p)}(t)| \le 1$ almost everywhere. For functions $U_f(r,\alpha)$ harmonic inside Q and corresponding to functions $f(t) \in W^{(p)}$, there holds

Theorem 2. Whatever the function $f(t) \in W^{(p)}$, for any values r_1, r_2 ($0 \le r_1, r_2 \le 1$) the inequality

$$\left|U_f(r_1,t) - U_f(r_2,t)\right| \le \begin{cases} \left|U_{\varphi_p}(r_1,0) - U_{\varphi_p}(r_2,0)\right|, & p \text{ even}, \\ \left|U_{\varphi_p}(r_1,\pi/2) - U_{\varphi_p}(r_2,\pi/2)\right|, & p \text{ odd}, \end{cases}$$

holds, where $\varphi_p(t)$ is that function from $W^{(p)}$ for which $\varphi_p^{(p)} = \varphi_0(t)$.

The indicated extremal role of the functions $\varphi_p(t)$ in the class $W^{(p)}$ is preserved also for differences along arcs concentric with Q of circles.

For an arbitrary natural value k, denote by

$$\Delta_h^k U_f(r,t) = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} U_f(r, t+\nu h)$$

the difference of the function $U_f(r,t)$ with respect to t of order k with step h.

Theorem 3. Whatever the function $f(t) \in W^{(p)}$ and the natural number $k \le p+1$, for any positive $h \le \pi$ everywhere in the disk Q the inequality

$$\left|\Delta_h^k U_f(r,t)\right| \le \begin{cases} \left|\Delta_h^k U_{\varphi_p}(r, -k(\pi+h/2))\right|, & p-k \text{ even}, \\ \left|\Delta_h^k U_{\varphi_p}(r, -k(\pi+h/2)+\pi/2)\right|, & p-k \text{ odd}. \end{cases} \tag{4}$$

Inequality (4) for any $0 \le r_1, r_2 < 1$ remains valid also when $p = 0,\ k = 1$ for every function $f(t)$ satisfying condition (1).

Difference inequalities with respect to the radius and with respect to circles analogous to those given above also hold for the harmonic functions $\widetilde{U}_f(r,t)$ conjugate to the functions $U_f(r,t)$, i.e., related to them by the Cauchy-Riemann equations.

Theorem 4. Whatever the function $f(t) \in W^{(p)}$, for any values r_1, r_2 $(0 \le r_1, r_2 \le 1)$ the inequality

$$\left|\widetilde{U}_f(r_1,t) - \widetilde{U}_f(r_2,t)\right| \le \begin{cases} \left|\widetilde{U}_{\varphi_p}(r_1,\pi/2) - \widetilde{U}_{\varphi_p}(r_2,\pi/2)\right|, & p \text{ even}, \\ \left|\widetilde{U}_{\varphi_p}(r_1,0) - \widetilde{U}_{\varphi_p}(r_2,0)\right|, & p \text{ odd}. \end{cases} \tag{5}$$

Inequality (5) for any r_1, r_2 $(0 \le r_1, r_2 < 1)$ remains valid also when $p = 0$ for every function $f(t)$ satisfying condition (1).

Theorem 5. Whatever the function $f(t) \in W^{(p)}$ and the natural number $k \ge p$, for any positive $h \le \pi$ everywhere in the disk Q the inequality

$$\left|\Delta_h^k \widetilde{U}_f(r,t)\right| \le \begin{cases} \left|\Delta_h^k \widetilde{U}_{\varphi_0}(r, -k(\pi+h/2))\right|, & p-k \text{ odd}, \\ \left|\Delta_h^k \widetilde{U}_{\varphi_p}(r, -k(\pi+h/2)+\pi/2)\right|, & p-k \text{ even}. \end{cases}$$

Replacing condition (1) by the condition

$$\int_0^{2\pi} |f(t)|\, dt \le 1, \tag{6}$$

and the class $W^{(p)}$ by the class $W^{(p)}L$ of all functions $f(t)$ having an absolutely continuous derivative of order $p-1$ and such that $\int_0^{2\pi} |f^{(p)}(t)|\, dt \le 1$, one can obtain sharp inequalities for the mean oscillations of the corresponding solutions of the Dirichlet problem along radii and along circles concentric with Q.

Theorem 6. If the point (r_1, r_2) belongs to the shaded region in Fig. 1, then

$$\sup_{\int_0^{2\pi} f(t)\, dt = 0,\ \int_0^{2\pi} |f(t)|\, dt \leqslant 1} \int_0^{2\pi} \left|U_f(r_1,t) - U_f(r_2,t)\right| dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^\nu \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{2\nu+1} \right|.$$

For all natural p and arbitrary values r_1, r_2 $(0 \leqslant r_1, r_2 \leqslant 1)$, the inequality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} \left| U_f(r_1,t) - U_f(r_2,t) \right| \, dt = \frac{4}{\pi} \sum_{\nu=0}^{\infty} (-1)^{\nu(p+1)} \left| \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{(2\nu+1)^{p+1}} \right|$$

holds.

Theorem 7. For all natural k and p $(k \leqslant p+1)$, for arbitrary values r and h $(0 \leqslant r \leqslant 1,\ 0 < h \leqslant \pi)$, the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} \left| \Delta_h^k U_f(r,t) \right| \, dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu(p+k+1)} \frac{\{2 \sin(\nu + 1/2)h\}^k}{(2\nu+1)^{p+1}} \, r^{2\nu+1} \right|$$

holds.

The equality is also valid in the case $p = 0,\ k = 1,\ 0 \leqslant r < 1$, if the class $W^{(p)}L$ is replaced by the class of all functions satisfying conditions (2) and (6).

Theorem 8. For all natural p and arbitrary values r_1, r_2 $(0 \leqslant r_1, r_2 \leqslant 1)$, the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} \left| \widetilde{U}_f(r_1,t) - \widetilde{U}_f(r_2,t) \right| \, dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu p} \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{(2\nu+1)^{p+1}} \right|$$

holds.

The last relation remains valid for $p = 0,\ 0 \leqslant r_1, r_2 < 1$, if the class $W^{(p)}L$ is replaced by the class of all functions satisfying conditions (2) and (6).

Theorem 9. For all natural p, k $(k \leqslant p)$, for all values r $(0 \leqslant r \leqslant 1)$ and $0 < h \leqslant \pi$, the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} \left| \Delta_h^k \widetilde{U}_f(r,t) \right| \, dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu(k+p)} \frac{\{2 \sin(\nu + 1/2)h\}^k}{(2\nu+1)^{p+1}} \, r^{2\nu+1} \right|.$$

The inequalities given here for oscillations of harmonic functions contain a number of different results obtained earlier in other works.

Theorem 1, for $r_2 = 0$, gives the above-mentioned Schwarz inequality for the maximum of the modulus of a harmonic function; and Theorem 4, when $p = 0$ and $r_2 = 0$, gives the well-known Köbe inequality ([2]) for the maximum of the modulus of a function harmonic in Q, whose conjugate is bounded and has boundary values satisfying conditions (1) and (2). Theorem 2, for $r_1 = 0,\ r_2 = 1$, gives an inequality equivalent to the known inequalities of Bohr (^{3}) (when $p = 1$) and S. N. Bernstein (^{4}) (when $p > 1$) for the maximum of the modulus of functions $f(t) \in W^{(p)}$ satisfying condition (2). For $r_2 = 1$, Theorem 2 gives the sharp estimate obtained in (^{7}) of the uniform deviation of a harmonic function from its boundary values. Theorem 6 for natural values of p in the case $r_2 = 1$ was obtained earlier in (^{8}) (Chapter VI, § 3, p. 155). Theorems 4 and 8, for $r_1 = 0$, $r_2 = 1$, give inequalities of N. I. Akhiezer and M. G. Krein (^{5}) (see also (^{6})) for the maximum of the modulus and the mean value of the modulus of functions conjugate to functions $f(t) \in W^{(p)}$.

A special case of Theorem 3, when $r = 1$, $k = 1$, gives an inequality for the modulus of continuity of functions $f(t) \in W^{(p)}$ equivalent to the result obtained in (9) (Theorem 1). For arbitrary k, in the case when $r = 1$, Theorems 3, 5, 7, and 9 give the sharp inequalities obtained in (10) for the moduli of smoothness of higher orders of functions $f(t) \in W^{(p)}$ and of their conjugates.

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Note: Figure translations are in progress. See original paper for figures.