# NONLINEAR EQUATIONS WITH MONOTONE AND OTHER OPERATORS

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## Abstract

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**MATHEMATICS**

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# NONLINEAR EQUATIONS WITH MONOTONE AND OTHER OPERATORS

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For nonlinear operator equations of the form $x = B(x)$ with a completely continuous operator $B(x)$, the convergence of the Galerkin method was proved by M. A. Krasnosel' skii ([1]). In the work of S. G. Mikhlin and L. N. Gaten-Torn ([4]), the solvability of Ritz systems was proved for equations of the form $F(x) = \theta$, where $F(x)$ is not necessarily compact. The present work continues the studies mentioned above. We prove the convergence of the Ritz-Galerkin method for the equation $F(x) = \theta$, where $F(x)$ need not contain a compact part. The results obtained cover the class of monotone operators introduced by the author in ([5]). Earlier, F. E. Browder, J. Leray, and J. Lions used the Galerkin method to prove the existence of a solution of equations with monotone operators. Among other works we mention ([7,8]). We do not touch upon the numerous works carried out in this direction for linear equations and for concrete nonlinear equations. Further, in § 4 one new theorem is obtained on the solvability of equations $F(x) = \theta$, generalizing in certain directions the second principle of Schauder. Finally, a further generalization of the notion of a nonlinear monotone operator is given and a number of its properties are studied.

1. In §§ 1-3, $E$ is a real separable reflexive Banach space; $E^*$ is its conjugate; $M$ is a linear everywhere dense set in $E$; linear combinations of the elements $x_i \in M$ $(i = 1, 2, \dots)$ lie densely in $E$, and any finite number of $x_i$ are linearly independent; $F(x)$ is a linear or nonlinear operator from $E$ into $E^*$ (sometimes $F(x)$ is defined only on $M$). For $z \in E^*$ and $x \in E$, $(z, x)$ is the value of the linear functional $z$ on the element $x$. We shall seek approximate solutions of the equation $F(x) = \theta$ by the Ritz-Galerkin method. As shown in ([4]), this leads to the question of solvability of the Ritz-Galerkin systems

$$\left(F\left(\sum_{i=1}^{n} a_i x_i\right), x_j\right) = 0 \quad (j = 1, \dots, n), \tag{1}$$

where $a_1, \dots, a_n$ are real numbers.

**Theorem 1** (cf. [4]). *Let $F(x)$ from $M$ into $E^*$ be such that each function*

$$\varphi_j = \left(F\left(\sum_{i=1}^{n} a_i x_i\right), x_j\right)$$

*of $n$ variables $a_1, \dots, a_n$ $(j = 1, \dots, n)$ is continuous for every natural $n$. Suppose that for all $x \in M \cap S$, where $S = \{x : \|x\| = R > 0,\ x \in E\}$, the condition $(F(x), x) \geq 0$ is fulfilled. Then system (1) is solvable for every natural $n$.*

2. In this section it is assumed that $F(x)$ is defined on all of $E$. Denote

$$a_0^n = \sum_{i=1}^{n} a_i^0 x_i,$$

where $(a_1^0, \dots, a_n^0)$ is some solution of system (1). The element $a_0^n$ is called a Ritz-Galerkin approximation.

**Lemma 1.** *Suppose that for $M = E$ all the conditions of Theorem 1 are fulfilled. Suppose $F(x)$ is bounded in the ball $\|x\| \leq R$. Then*

$$F(a_0^n) \xrightarrow[n\to\infty]{\text{weakly}} \theta.$$

Suppose now that $F'(x)$ is a potential operator and the functional $f(x)$ is its potential, i.e. $\operatorname{grad} f(x) = F(x)$.

**Lemma 2.** If $f(x)$ is continuous in $E$, weakly lower semicontinuous in every ball of $E$, and $\lim_{\|x\|\to\infty} f(x) = +\infty$, then $\inf_{x\in E} f(x) > -\infty$, and the Ritz-Galerkin approximations $a_0^n$ form a minimizing sequence for $f(x)$, i.e.

$$\lim_{n\to\infty} f(a_0^n) = \inf_{x\in E} f(x)^*.$$

**Theorem 2.** Let the functional $f(x)$ satisfy the conditions of Lemma 2 and, in addition, let $\inf_{x\in E} f(x)$ be attained at no more than one point. Then there exists $x_0 \in E$ such that

$$\inf_{x\in E} f(x) = f(x_0)$$

and $a_0^n \xrightarrow[n\to\infty]{\text{weakly}} x_0$.

**Corollary 1.** A strictly monotone (i.e. $(F(x_2) - F(x), x_2 - x_1) > 0$ for all $x_2 \neq x_1$ in $E$) operator $F(x)$ with continuous potential, satisfying the condition

$$\lim_{\|x\|\to\infty} \int_0^1 (F(tx), x)\, dt = +\infty,$$

has a unique point $x_0 \in E$, $F(x_0) = \theta$, to which the approximations $a_0^n$ converge weakly.

**Theorem 3.** Let the functional $f(x)$ be continuous and Gâteaux differentiable on all of $E$, $\operatorname{grad} f(x) = F(x)$, let $F(x)$ be continuous along each ray, and

$$(F(x_2) - F(x_1), x_2 - x_1) \geq \gamma(\|x_2 - x_1\|)\|x_2 - x_1\|, \tag{2}$$

for all $x_2, x_1 \in E$, where $\gamma(t) \geq 0$ for $t \geq 0$,

$$\nu(R) = \int_0^1 \gamma(tR)\, dt;$$

$\nu(R)$ is continuous for $R \geq 0$; $\nu(0) = 0$; $\nu(R) > 0$ for $R > 0$; $\lim_{R\to\infty} \nu(R) = +\infty$. Then $f(x)$ has a unique point of minimum $x_0 \in E$, to which the approximations $a_0^n$ converge strongly.

Let us also note that, for solvability of system (1), the following conditions are sufficient: $f(x)$ is lower semicontinuous in any ball of $E$ and

$$\lim_{\|x\|\to\infty} \int_0^1 (F(tx), x)\, dt = +\infty.$$

**3.** If the operator $F(x)$ is not potential, then under certain conditions the Galerkin method also converges.

**Theorem 4.** Let $F(x)$ be defined on all of $E$, satisfying the conditions: a) the operator $F(x)$ is finitely continuous**; b)

$$(F(x_2) - F(x_1), x_2 - x_1) \geq \gamma(\|x_2 - x_1\|)\|x_2 - x_1\|$$

for all $x_2, x_1 \in E$, where $\gamma(t) \geq 0$ for $t \geq 0$, $\gamma(t)$ is continuous and increasing, $\lim_{t\to\infty} \gamma(t) = +\infty$; c) for some $R > 0$ such that $\gamma(R) \geq \|F(\theta)\|$, the operator $F(x)$ is bounded in the ball $\|x\| \leq R$. Then the approximations $a_0^n$ converge strongly to the unique solution $x^*$ of the equation $F(x) = \theta$.

**4. Theorem 5.** Let $E$ be a real separable reflexive Banach space; let $D$ be a bounded convex closed set in $E$, containing zero as an interior point. Let a linear or nonlinear operator $F(x)$, acting from $D$ into $E^*$, be weakly continuous in $D$ (i.e. if $x_n \in D$, $x_n \xrightarrow{\text{weakly}} x_0$, then $F(x_n) \xrightarrow{\text{weakly}} F(x_0)$ in $E^*$) and let it satisfy on the boundary of $D$ the condition $(F(x), x) \geq 0$. Then the equation $F(x) = \theta$ has at least one solution in $D$.

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* A statement of this type is contained in the theses of a report by S. G. Mikhlin at the All-Union Conference on the Application of Methods of Functional Analysis, Baku, 1965, and in other works.

** That is, for any finite-dimensional subspace $E^m \subset E$, from the fact that $x_k, x_0 \in E^m$, $\lim_{k\to\infty} \|x_k - x_0\|_E = 0$, it follows that

$$F(x_k) \xrightarrow[k\to\infty]{\text{weakly}} F(x_0) \quad \text{in } E^*.$$

From this theorem, in particular, there follows Schauder' s second principle for separable Hilbert spaces.

5. Let $X$ and $Y$ be two groups, and let $Z$ be a linear partially ordered vector space. Suppose that for all $x \in X$, $y \in Y$ there is defined an operation of multiplication $xy$, with values of the product in $Z$ (it is assumed that $x(y_1 + y_2) = xy_1 + xy_2$ and $(x_1 + x_2)y = x_1 y + x_2 y$ for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$).

**Definition.** A set $\Omega \subset X \times Y$ will be called **monotone** if, for any $\{x_1, y_1\}, \{x_2, y_2\} \in \Omega$,

$$(x_1 - x_2)(y_1 - y_2) \geq \theta_Z,$$

and **maximal monotone** if it is monotone and is not properly contained in any other monotone set. An operator $F(x)$ (linear or nonlinear) from $D \subset X$ into $Y$ is monotone if its graph in $D$ is a monotone set, i.e., if

$$(x_2 - x_1)[F(x_2) - F(x_1)] \geq \theta_Z$$

for all $x_2, x_1 \in D$.

This definition generalizes the notion of monotonicity studied in recent years in a number of works (([5–6, 9–12]) and others). In what follows, $X$ and $Y$ are Banach spaces, $Z$ is a real Banach space with a closed cone,* and the product $xy$ is assumed to be continuous in the totality of the variables. The basic properties are as follows.

**Theorem 6.** If $\Omega \subset X \times Y$ is a maximal monotone set, then

$$W = \{x : \{x, y_0\} \in \Omega,\ y_0 \in Y\}$$

is convex and weakly closed. In particular, if the graph of an operator $F(x)$ is a maximal monotone set, then the set of all solutions of the equation $F(x) = y_0$ is convex and weakly closed for any $y_0 \in Y$.

**Theorem 7.** Let $F(x)$ be given on a convex set $D \subset X$. Each of the following conditions is necessary and sufficient for the monotonicity of $F(x)$ on $D$:
a) for any fixed $x$, $(x + h) \in D$, the abstract function of the real variable $t$,

$$\varphi_{x,h}(t) = hF(x + th),$$

has the property that, if $0 \leq t_1 \leq t_2 \leq 1$, then

$$\varphi_{x,h}(t_2) \geq \varphi_{x,h}(t_1);$$

b) for each point $x \in D$ there exists an open ball $S(x)$, $x \in S(x)$, such that $F(x)$ is monotone on $D \cap S(x)$;
c) if $D$ is open, $F(x)$ has a Gâteaux derivative in $D$, and the form $h[F'(x)h]$ is continuous with respect to $x \in D$, then for the monotonicity of $F(x)$ on $D$ it is necessary and sufficient that

$$h[F'(x)h] \geq \theta_Z$$

for all $x \in D,\ h \in X$.

We shall call an operator $F(x)$ **semicontinuous on the set** $D$ (cf. ([10])) if, for any $v, x \in X$, $u \in D$, and real $t$,

$$\lim \|x[F(u+tv) - F(u)]\| = 0.$$

**Theorem 8.** Suppose the product $xy$ is such that, for every $y_0 \neq \theta_Y$, there exists $x \in X$ such that $xy_0 \neq \theta_Z$. Then the graph of every semicontinuous maximal** monotone operator defined on an open set $D \subset X$ is a maximal monotone set.

**Theorem 9.** Suppose that, for every $\bar{y}_0 \in Y$, $\bar{y}_0 \neq \theta_Y$, there exists $x \in X$ such that

$$x\bar{y}_0 \geq \theta_Z, \qquad x\bar{y}_0 \neq \theta_Z.$$

Let $D$ be an open set, $x_0 \in D$, $w_0 \in Y$, and

$$(x - x_0)[F(x) - w_0] \geq \theta_Z$$

for all $x \in D$. If the operator $F(x)$ is semicontinuous, then $w_0 = F(x_0)$.

**Theorem 10.** Let $X$ be a reflexive Banach space and suppose that, for every $y_0 \neq \theta_Y$, there exists $x \in X$ such that $xy_0 \neq \theta_Z$. Suppose that on a bounded convex closed set $D \subset X$ there is given an operator $F(x)$ (not necessarily monotone) with values in $Y$ such that, for any $v \in X$, from the fact that $x_n \in D$, $x_n \rightharpoonup x_0$, it follows that

$$\lim_{n\to\infty} \|v[F(x_n) - F(x_0)]\|_Z = 0.$$

Suppose that every finite system of inequalities

$$v_j F\left(\sum_{i=1}^{n} a_i v_i\right) \geq \theta_Z$$

with

* A set $K \subset Z$ is a closed cone if $K$ is closed, convex, and if $x \in K$, $\lambda \geq 0$, then $\lambda x \in K$; while for $x \in K$, $x \neq 0$, $(-x) \notin K$.

** An operator is called maximal monotone if it has no proper monotone extension.

for every natural $n$ has at least one solution $(a_1, \dots, a_n)$,

$$\sum_{i=1}^{n} a_i v_i \in D, \quad (j = 1, \dots, n),$$

where $a_i$ are real numbers, for arbitrary elements $v_i \in X$. Then the equation $F(x) = \theta_Y$ has a solution in $D$.

**Lemma 3.** If the graph of the operator $F(x)$ is a maximal monotone set, then $F(x)$ is a closed operator.

We note that some of the properties valid for monotone operators are true for operators of a more general type, defined on arbitrary sets (not necessarily from a group). A number of properties given in Theorems 6–9 were obtained, for the scalar case, by F. E. Browder, G. J. Minty, and the author.

The theorems on the convergence of the Galerkin method admit applications to various problems of analysis. An estimate of the rate of convergence has been obtained.

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*Note: Figure translations are in progress. See original paper for figures.*