ON PRINCIPAL INVOMORPHISMS OF LIE ALGEBRAS

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Abstract

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ON PRINCIPAL INVOMORPHISMS OF LIE ALGEBRAS

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In this note it will be shown that a simple semisimple compact Lie algebra over a field R has a principal invomorphism. This result is obtained using the apparatus of involutive sums ([1]), bypassing H. Weyl' s root method, which gives a new approach to the theory of simple Lie algebras.

1. Everywhere in what follows we consider Lie algebras over the field R. Denote the connected adjoint group of a Lie algebra Γ by $\mathrm{Int}(\Gamma)$. The restriction of $\mathrm{Int}(\Gamma)$ to a subalgebra $Q \subseteq \Gamma$ will be denoted by $\mathrm{Int}_\Gamma(Q)$, so that $\mathrm{Int}_\Gamma(\Gamma) = \mathrm{Int}(\Gamma)$. The restriction of $\mathrm{Int}_\Gamma(Q)$ to some invariant subspace $K \subseteq \Gamma$ will be denoted by $\mathrm{Int}_\Gamma^K(Q)$. An automorphism A of the algebra Γ such that $A^2 = I$ will be called an **invomorphism**; the maximal set of vectors $y \in \Gamma$ such that $Ay = y$ forms a subalgebra $L \subseteq \Gamma$, which we shall call the **invoalgebra** of the invomorphism A, and the pair Γ/L will be called an **invopair**.

Definition 1. An invomorphism A of a Lie algebra P will be called **principal** if its invoalgebra L has a simple three-dimensional ideal B_3. Accordingly, we shall then say that L is a **principal invoalgebra**, and P/L a **principal invopair**.

Definition 2. Let A be a principal invomorphism of a Lie algebra P, and let B_3 be a simple three-dimensional ideal of its invoalgebra L. Then A will be called **principal orthogonal** (or of **type** O) if $\mathrm{Int}_P(B_3) = SO(3)$, and **principal unitary** (or of **type** U) if $\mathrm{Int}_P(B_3) = SU(2)$. Accordingly, we shall distinguish orthogonal and unitary principal invoalgebras L and principal invopairs P/L.

We note that if A is a principal invomorphism in P, then $\mathrm{Int}_P(B_3)$ is a three-dimensional simple compact connected Lie group and therefore is isomorphic either to $SO(3)$ or to $SU(2)$, as is well known.

Definition 3. An invomorphism A of an algebra P will be called **central** if its invoalgebra L has a nontrivial center. Accordingly, we shall speak of a **central invoalgebra** and **invopair**.

Definition 4. A principal invomorphism A of a Lie algebra P will be called **principal biunitary** (or of **type** U^2) if its invoalgebra $L = P_3 \oplus Q_3 \oplus \bar{L}$, where $\mathrm{Int}_P(P_3) = SU(2)$, $\mathrm{Int}_P(Q_3) = SU(2)$. Accordingly, we shall speak of a **principal biunitary invoalgebra** and **invopair**.

Definition 5. A unitary but not biunitary principal invomorphism will be called **principal monounitary** (or of **type** $U^{(1)}$). Accordingly, we shall speak of a **principal monounitary invoalgebra** and **invopair**.

Definition 6. An invomorphism A of a Lie algebra P will be called **special** if its invoalgebra L has a principal invomorphism of type O. Accordingly, we shall then say that L is a **special invoalgebra**, and P/L a **special invopair**.

Definition 7. Let L be a special invoalgebra of a Lie algebra P, and let B_3 be a three-dimensional simple ideal of its principal invoalgebra; we shall say that the invoalgebra L is an **orthogonal special** (or of type ···

O), if $\mathrm{Int}_P(B_3) = SO(3)$, and **unitary special** (or of **type** U), if $\mathrm{Int}_P(B_3) = SU(2)$. Accordingly, we shall distinguish **orthogonal** and **unitary special invomorphisms** and **invopairs**.

2. We describe the construction of the superinvolutive decomposition of a Lie algebra. Let Γ be simple and compact, and let B_3 be its three-dimensional simple subalgebra, with $\mathrm{Int}_\Gamma(B_3) = SO(3)$. In $SO(3)$ one can choose elements $S_1, S_2, S_3, p, \varphi_1, \varphi_2, \varphi_3$ such that

$$(p)^3 = (S_\rho)^2 = I, \quad S_\rho \neq I, \quad S_\rho S_\mu = S_\mu S_\rho \quad (\rho, \mu = 1, 2, 3), \quad S_1 S_2 = S_3,$$

$$S_2 S_3 = S_1, \quad S_3 S_1 = S_2, \quad p S_1 = S_2 p, \quad p S_2 = S_3 p, \quad p S_3 = S_1 p; \qquad (1)$$

$$(\varphi_\rho)^2 = S_\rho \quad (\rho = 1, 2, 3), \quad \varphi_1 \varphi_2 = \varphi_2 \varphi_3 = \varphi_3 \varphi_1 = p. \qquad (1')$$

Then in $\mathrm{Int}_\Gamma(B_3)$, too, (1) and (1′) will hold by virtue of the isomorphism $\mathrm{Int}_\Gamma(B_3) = SO(3)$. Let L_1, L_2, L_3 be the invoalgebras of the invomorphisms S_1, S_2, S_3, respectively. From (1) there then follows the involutive decomposition

$$\Gamma = L_1 + L_2 + L_3, \quad L_1 \cap L_2 = L_2 \cap L_3 = L_3 \cap L_1 = L_0, \quad L_\rho / L_0 \text{ are invopairs}, \qquad (2)$$

$$L_\rho = E_\rho + L_0, \quad \text{where } E_\rho \perp L_0 \text{ in } \Gamma \ (\rho = 1, 2, 3), \quad pE_1 = E_2, \quad pE_2 = E_3,$$

$$pE_3 = E_1, \quad pL_0 = L_0.$$

Definition 8. If in $\mathrm{Int}(\Gamma)$ there exist automorphisms satisfying relation (1), then the decomposition (2) will be called **superinvolutive**.

Definition 9. We shall call a superinvolutive decomposition **simple** if the restriction of the automorphism p from (1) to L_0 is the identity automorphism, and **general** otherwise.

Definition 10. A superinvolutive decomposition of the algebra Γ will be called **principal** for a principal invomorphism S of type O with invoalgebra $L = B_3 \oplus \widetilde{L}$, if the invomorphisms S_1, S_2, S_3 of the superinvolutive decomposition belong to $\mathrm{Int}_\Gamma(B_3)$, $pS = Sp$, and $px = x$ for $x \in \widetilde{L}$.

Theorem 1. Let Γ be a simple compact Lie algebra; $L = B_3 \oplus \widetilde{L}$ the principal invoalgebra of an invomorphism S of type O; then, if Γ admits a simple superinvolutive decomposition principal for S, with invoalgebras L_1, L_2, L_3, then

$$\Gamma/L = so(m)/(so(m-3) \oplus so(3)) \quad (m \neq 4, 2, 1);$$

$$\Gamma/L_\rho = so(m)/(so(m-2) \oplus so(2)) \quad (\rho = 1, 2, 3,\ m \neq 4, 2, 1)$$

(with natural embeddings).

Theorem 2. Let Γ be simple and compact; $L = B_3 \oplus \widetilde{L}$ the principal invoalgebra of an invomorphism S of type O; $\widetilde{L} \neq \{0\}$; then

$$\Gamma/L = so(m)/(so(m-3) \oplus so(3)) \quad (m > 4)$$

with the natural embedding.

Theorem 3. Let Γ be simple and compact, $L = B_3 \oplus \widetilde{L}$ the principal invoalgebra of an invomorphism S of type O; Γ does not admit a simple superinvolutive decomposition principal for S; then $B_3 = L$, $\dim \Gamma = 8$, and

$$\Gamma/L = su(3)/so(3)$$

with the natural embedding.

3. Theorems 1, 2, 3, together with the construction of the associated invomorphism, make it possible to clarify the structure of special unitary invomorphisms and to prove below the existence of a principal invomorphism for any simple semisimple compact Lie algebra.

Definition 11. Let Γ be a compact Lie algebra; $L = B_3 \oplus \widetilde{L}$ an invoalgebra of an invomorphism A of type O; $M = P_3 \oplus Q_3 \oplus \widetilde{M}$ an invoalgebra of a biunitary principal invomorphism J; $JA = AJ$; B_3 the diagonal in $P_3 \oplus Q_3$ of the canonical invomorphism $P_3 \leftrightarrow Q_3$; then we shall say that J is an **associated**

invomorphism for A; correspondingly, we shall say that M is an **associated invoalgebra** for L, and that Γ/M is an **associated invopair** for Γ/L.

Theorem 4. Let Γ be simple and compact; let $\Gamma/(B_3 \oplus L)$ be the principal invariant of type O of an invomorphism A; $L \neq \{0\}$; then there exists in Γ an invomorphism J associated with A.

Theorem 5. If a simple compact algebra Γ admits an orthogonal principal invomorphism $A \neq I$, then Γ also admits a unitary principal invomorphism J, with $JA = AJ$.

Theorem 6. Let Γ be simple and compact, and let some invosubalgebra of it for an invomorphism $A \neq I$ admit a principal invomorphism; then Γ admits a unitary special invomorphism.

Theorem 7. If a simple compact Lie algebra has a unitary special invosubalgebra, then it also has a principal invosubalgebra.

Theorems 6 and 7 lead, by induction, to Theorem 8.

Theorem 8. A simple and semisimple compact Lie algebra has a principal invomorphism.

From Theorems 8 and 5 it follows:

Theorem 9. If Γ is a simple compact noncommutative Lie algebra and $\dim \Gamma \neq 3$, then Γ has a unitary principal invomorphism.

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CITED LITERATURE

1. L. V. Sabinin, *DAN*, **165**, No. 5 (1965).

Note: Figure translations are in progress. See original paper for figures.