ON THE ORDER OF GROWTH OF LEBESGUE FUNCTIONS OF BOUNDED ORTHONORMAL SYSTEMS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.512.7

MATHEMATICS

A. M. OLEVSKII

ON THE ORDER OF GROWTH OF LEBESGUE FUNCTIONS OF BOUNDED ORTHONORMAL SYSTEMS

(Presented by Academician A. N. Kolmogorov on 28 XII 1966)

§ 1. In a recent paper ([1]) we established a theorem on the impossibility of constructing an orthonormal system of uniformly bounded functions such that every continuous function would be representable by a Fourier series converging uniformly (or even simply at every point). This theorem was obtained as a result of studying the properties of Lebesgue functions of bounded systems. Namely, it turned out that if the system is bounded in the aggregate, i.e.

$$|\varphi_n(x)| < M \qquad (0 \leq x \leq 1;\ n = 1, 2, ...), \tag{1}$$

then the Lebesgue functions of this system

$$L_n(x) = \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(x)\varphi_k(t) \right| dt$$

cannot be bounded in the aggregate; moreover, at every point x of some set of positive measure the relation $\lim\limits_{n\to\infty} L_n(x) = \infty$ holds. In connection with this result the question naturally arose about the order of growth of the Lebesgue functions. Do there exist uniformly bounded orthonormal systems with Lebesgue functions growing arbitrarily slowly, or does condition (1) predetermine the presence of some fixed growth of the Lebesgue functions? In the latter case the question consists in estimating this growth.

It is easy to indicate an equivalent formulation of the question in terms of Fourier series.

The main result of the present note is the following assertion, which gives an answer to this question. It turns out that the condition of uniform boundedness of the system inevitably entails at least logarithmic growth of the Lebesgue functions. More precisely, the following holds.

Theorem 1. *Let $\{\varphi_n(x)\}$ be an arbitrary orthonormal system in $L^2[0,1]$ satisfying condition (1). Then the relation*

$$\overline{\lim_{n\to\infty}} \frac{L_n(x)}{\ln n} > 0, \qquad x \in E \subset [0,1],\ \mu E > 0. \tag{2}$$

holds.

The sharpness of this estimate is obvious if one takes into account that in the classical case of the trigonometric system we have $L_n \sim C \ln n$. Let us note further that the upper limit here cannot be replaced by the lower one: the Lebesgue functions need not grow for all indices.

Finally, even under the additional condition of completeness of the system $\{\varphi_n\}$, relation (2) need not hold almost everywhere: there exists a complete orthonormal uniformly bounded system $\{\Psi_n(x)\}$ on $[0,1]$,

for which $L_n(x) < C$ for $x \in [0,\gamma]$, $n = 1,2,...$, where γ is any preassigned number, $0 < \gamma < 1$ (see (1), Theorem 3).

Theorem 1 can be reformulated in an equivalent way in terms of Fourier series of continuous functions. In this form we obtain an estimate, sharp in order, for the divergence of such series. Namely, the following holds.

Theorem 2. *Let $\{\varphi_n\}$ be an orthonormal system satisfying condition (1). Then for every positive sequence $\omega(n) = o(\ln n)$ there exists a continuous function $f(x)$ whose Fourier partial sums $S_n(f;x)$ at some point x_0 satisfy the condition*

$$\overline{\lim_{n\to\infty}} \frac{S_n(f;x_0)}{\omega(n)} = \infty. \tag{3}$$

Thus, for any bounded system, Fourier series of continuous functions may diverge quite rapidly, namely, not more slowly than any sequence $\omega(n) = o(\ln n)$. At the same time, already for the trigonometric system it is known that $S_n(f;x) = o(\ln n)$.

Analogously to (1), one can construct a function $f(x)$ for which relation (3) holds on a sufficiently dense set (of cardinality continuum, of the second category on some perfect set of positive measure). At the same time, not every point of the interval can occur as x_0 in (3), in contrast to the case of the trigonometric system. In particular, the system $\{\Psi_n\}$ mentioned above is in a certain sense better than the trigonometric one: for it, the Fourier series of all continuous functions converge uniformly outside the interval $[\gamma,1]$, which can be made arbitrarily small. Let us further note that a consequence analogous to Theorem 2 can also be obtained for Fourier series in L. The rate of divergence of such series in the L-metric is estimated by the following theorem.

Theorem 3. *For every uniformly bounded orthonormal system* $\{\varphi_n\}$ *and for every positive sequence* $\omega_n = o(\ln n)$ *there exists a function* $f(x)$ *for which*

$$\overline{\lim_{n\to\infty}} \frac{1}{\omega(n)} \|S_n(f;x)\|_L = \infty.$$

§ 2. The results formulated above follow easily from the following general inequality.

Main inequality. *Let* $\{\varphi_k(x)\}$ $(1 \le k \le n)$ *be an orthonormal system in* $L^2[0,1]$, *and let* $\{c_k\}$ *be arbitrary numbers, with the conditions*

$$|\varphi_k(x)| < M, \qquad |c_k| < M \qquad (1 \le k \le n,\ 0 \le x \le 1).$$

Then the inequality holds

$$\max_{1\le\nu\le n} \int_0^1 \left| \sum_{k=1}^{\nu} c_k \varphi_k(x) \right| dx \ge C(M) \frac{\ln n}{n} \sum_{k=1}^{n} c_k^2, \tag{4}$$

where $C(M) > 0$ *is a constant depending only on* M, *and not on* $\{\varphi_k\}$, $\{c_k\}$, *or* n.

The method of proof of this inequality is a development of the method applied in (1). Without discussing here the proof of (4), we indicate some further consequences.

A consequence relating to Littlewood' s hypothesis. The indicated hypothesis (2) essentially concerns the behavior of the Lebesgue constants of rearranged trigonometric systems. It can be formulated as follows: for every permutation of the natural sequence $\{\nu_k\}$ the relation

$$\lim_{n\to\infty} \frac{I_n}{\ln n} > 0, \qquad \text{where } I_n = \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} e^{i\nu_k x} \right| dx.$$

This conjecture has been the subject of investigation by a number of authors ([3–6]). In particular, Salem ([3]) established, under special assumptions concerning the sequence $\{v_k\}$, the relation

$$\overline{\lim_{n\to\infty}} \frac{I_n}{(\ln n)^{1/2}} > 0.$$

An important advance was achieved by Cohen ([4]): for any permutation $\{v_k\}$,

$$\overline{\lim_{n\to\infty}} \frac{I_n}{(\ln n/\ln\ln n)^{1/8}} > 0. \tag{5}$$

Slightly supplementing Cohen' s method, Davenport ([5]) replaced the exponent 1/8 in (5) by 1/4. Our result, which follows immediately from inequality (4), is the following.

Corollary. *For any uniformly bounded orthonormal system* $\{\varphi_n(x)\}$ *in* $L^2[0,1]$*, the relation*

$$\varlimsup_{n\to\infty} \frac{1}{\ln n} \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(x) \right| dx > 0 \tag{6}$$

holds.

Thus Littlewood' s conjecture is valid for any bounded system, but with the replacement of the lower limit by the upper limit, which is unavoidable in the class of such systems.

Corollary on series with monotone coefficients. In ([1]) the following generalization of Sidon' s theorem was proved: if an orthonormal system $\{\varphi_n(x)\}$, bounded in the aggregate, is complete in the space L, then not every series $\sum c_k\varphi_k(x)$, $c_k \downarrow 0$, is a Fourier-Lebesgue series. Inequality (4) makes it possible to strengthen this result.

Theorem 4. *Let* $\{\varphi_n(x)\}$ *be a complete orthonormal system in* $L[0,1]$ *satisfying condition (1). Then there exists a series*

$$\sum_{k=1}^{\infty} c_k\varphi_k(x), \qquad c_k = o\left(\frac{1}{\ln k}\right),$$

which is not the Fourier series of any function $f \in L$.

§ 3. In this paragraph we formulate some further results established with the aid of the basic inequality (4) and adjoining the theorem of the paper ([1]) cited at the beginning of the note.

On subsequences of convergence. Let $\{\varphi_n\}$ be a uniformly bounded orthonormal system. Then, according to the theorem mentioned above, there exists a continuous function with a divergent Fourier series with respect to this system. At the same time the system $\{\varphi_n\}$ may have a subsequence of convergence, i.e., such a fixed sequence of indices $\{n_k\}$ that, for every continuous function $f(x)$, the partial sums of the Fourier series $S_{n_k}(x)$ converge uniformly to $f(x)$. This means that the Lebesgue functions $L_n(x)$ along the indices n_k remain uniformly bounded.

It turns out, however, that there cannot be too many such indices. Roughly speaking, the sequence $\{n_k\}$ must grow in a geometric progression. More precisely, the following holds.

Theorem 5. *Let $\{\varphi_n(x)\}$ be an orthonormal system satisfying condition (1). Then, for every function $v(x)$, $v'(x) = o(v(x))$ $(x \to \infty)$, there exists a continuous function $f(x)$ for which the partial sums $S_{n_k}(x)$, $(n_k = [v(k)])$, diverge at some point.*

In particular, the sequence $n_k = k^s$ (s fixed), and even $n_k = [2^{k^\alpha}]$ $(\alpha < 1)$, cannot be a subsequence of convergence.

of a bounded system. At the same time $n_k = 2^k$ is a subsequence of convergence for the Walsh system.

On slowly increasing orthonormal systems. According to (1), if an orthonormal system $\{\varphi_n\}$ forms a basis in the space C, then $\|\varphi_n\|_C$ inevitably increases. The question arises how rapid this growth must be. Suppose the system $\{\varphi_n\}$ satisfies the condition

$$|\varphi_n(x)| < M_n \qquad (x \in [0,1];\ n = 1, 2, ...). \tag{7}$$

Can one, for a sufficiently slow growth of the numbers M_n, establish a theorem on the divergence of a Fourier series of type (1), or, however slowly the sequence M_n may grow, can one construct a system $\{\varphi_n\}$ satisfying condition (7) with respect to which every continuous function is expanded in a Fourier series converging to it uniformly (or simply everywhere)?

A curious feature arising here, in contrast to the case of bounded systems, is that the answers to these two questions are different.

Namely, if convergence everywhere is under discussion, then it is not difficult to prove the following assertion.

Theorem 6. *For any sequence $\{M_n \uparrow \infty\}$ one can construct an orthonormal system $\{\varphi_n\}$ satisfying condition (7), with respect to which every continuous function is represented by a Fourier series converging to it everywhere.*

(Of course, the Lebesgue functions of this system satisfy the condition $L_n(x) < C(x)$, $n = 1, 2, ...$, so that theorems on the growth of these functions of type (2) do not extend to increasing, even very slowly increasing, orthonormal systems.)

At the same time, in the question of uniform convergence of Fourier series (i.e. uniform boundedness of the Lebesgue functions), there is a certain stability: the nature of the result does not change when passing to the class of slowly increasing systems.

Theorem 7. *There exists a sequence $M_n \uparrow \infty$ having the property that no orthonormal system $\{\varphi_n\}$ satisfying condition (7) forms a basis in C.*

Thus, if $\{\varphi_n\}$ is a basis in C, then $\|\varphi_n\|_C$ must increase sufficiently rapidly. An interesting question, which for now remains open, consists in giving an exact estimate of this growth. Let us recall that for the classical basis—the Haar system—we have $\|\chi_n\|_C = O(\sqrt{n})$.

In conclusion we note one result concerning the relation between Lebesgue functions and Fourier series.

Theorem 2. *There exists a complete orthonormal system* $\{\varphi_n(x)\}$*, which is a system of convergence in L (i.e. every function* $f \in L$ *is expanded in an almost everywhere convergent Fourier series), and the Lebesgue functions of the system satisfy the condition* $\lim_{n\to\infty} L_n(x) = \infty$ *almost everywhere.*

Moscow Institute of Electronic
Machine Building

Received
22 XII 1966

REFERENCES

1. A. M. Olevskii, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **30**, No. 2 (1966).
2. G. H. Hardy, J. E. Littlewood, *J. London Math. Soc.*, **23**, Part 3, No. 94, 163 (1948).
3. R. Salem, *Amer. J. Math.*, **77**, No. 3, 535 (1955).
4. P. Cohen, *Amer. J. Math.*, **82**, No. 2 (1960).
5. H. Davenport, *Mathematica*, **7**, No. 14 (1960).
6. S. Saburo, *C. R.*, **260**, No. 10, 2673 (1965).

Note: Figure translations are in progress. See original paper for figures.