APPLICATION OF THE BHATNAGAR, GROSS, AND KROOK METHOD TO DETERMINING THE VELOCITY OF THERMAL SLIP OF A GAS NEAR A SOLID SURFACE

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Abstract

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APPLICATION OF THE BHATNAGAR, GROSS, AND KROOK METHOD TO DETERMINING THE VELOCITY OF THERMAL SLIP OF A GAS NEAR A SOLID SURFACE

As is well known, the thermal slip of a gas relative to a solid wall was first considered by Maxwell (^{1}), who gave the formula:

$$u = {}^3/_4 \nu \, d \ln T / dy, \tag{1}$$

where ν is the kinematic viscosity, T is the absolute temperature, and y is the coordinate along the wall.

However, Maxwell' s approach to the solution of this problem was rather crude, since it was based on the unjustified assumption that, before striking the wall surface, the molecules have the same velocity distribution as in the bulk of the gas. In reality, before striking the surface of a solid body, gas molecules undergo, on average, no fewer than one collision with molecules that have flown off from the surface of the solid body and that have (under complete accommodation) a normal Maxwellian velocity distribution, in contrast to the modified distribution of molecular velocities characteristic of the gas bulk in the presence of a temperature gradient.

As a result, in the Knudsen layer, of the order of the mean free path λ, a continuous transition must take place between the velocity distribution in the bulk and that directly at the wall surface. Therefore, before calculating the tangential momentum brought by gas molecules incident on the wall, it is necessary to determine how their velocity distribution changes in the Knudsen layer as a function of the distance from the wall in the presence of a tangential temperature gradient.

It is easy to see that the simplification introduced by Maxwell should overestimate the value of the slip coefficient. In a paper by one of us (^{2}), an attempt was made to estimate the slip coefficient by applying the Onsager principle to the Knudsen layer. However, the applicability of Onsager' s reciprocity principle in isolation to the Knudsen layer cannot be rigorously justified, and the results of the calculation apparently give a sharply underestimated value of the slip coefficient. Thus the question remains open.

In the present work an attempt is made to determine the slip velocity on the basis of the distribution function at the surface of a solid wall, obtained by the well-known method of Bhatnagar, Gross, and Krook (^{3}), since, apart from the noted lack of rigor in Maxwell' s approach, the thermodynamic approach may be questionable because of the use in the Knudsen layer of the differential form of the thermodynamic heat flux.

Let us consider a gas situated in a temperature-gradient field tangential to an infinite wall. We choose the origin of coordinates on the surface of the wall. The x-axis is directed perpendicular to the wall, and the y-axis along the surface. The distribution function of gas molecules over velocities depends, generally speaking, on the coordinates x and y, and can be found from the Boltzmann kinetic equation under stationary conditions,

$$(\mathbf{v}\nabla)f = (\partial f/\partial t)_{\text{coll}}, \tag{2}$$

where $\mathbf{v}$ is the velocity of a molecule; $f = f(x, y, \mathbf{v})$ is the distribution function;

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \int d\mathbf{v}_1 \int gb\, db\, d\varepsilon\, (f_1' f' - f_1 f)\,, \tag{3}$$

where $g = |\mathbf{v} - \mathbf{v}_1|$ is the relative velocity of two colliding molecules; b is the impact parameter of the collision; ε is the azimuthal scattering angle.

Let us estimate the order of magnitude of the collision term. We shall seek the distribution function in the form

$$f(x, y, \mathbf{v}) = f_u^{(0)}(y, \mathbf{v}) + \varphi(x, y, \mathbf{v}), \tag{4}$$

where $f_u^{(0)}$ is the local Maxwellian distribution function for a gas moving with mass velocity u; $\varphi(x, y, \mathbf{v})$ is a correction to the distribution function. Substituting (4) into (3), we obtain

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \int d\mathbf{v}_1 \int gb\, db\, d\varepsilon \left(f_{1u}'^{(0)}\varphi' - \varphi f_{1u}^{(0)} + f_u'^{(0)}\varphi_1' - \varphi_1 f_u^{(0)}\right). \tag{5}$$

All four terms under the integral sign have the same order. Let us estimate the integral of the second term in expression (5):

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \sim -\varphi \int d\mathbf{v}_1 \int gb\, db\, d\varepsilon f_{1u}^{(0)} = -\frac{\varphi}{\tau(\mathbf{v})},$$

where $\tau(\mathbf{v})$ has the order of magnitude of the free path time. τ, generally speaking, is a function of velocity. Thus, if the collision term is replaced by the expression

$$(\partial f/\partial t)_{\text{coll}} = -[f - f_u^{(0)}]/\tau(\mathbf{v}), \tag{6}$$

then the Boltzmann equation is substantially simplified. This substitution is precisely the method of Bhatnagar, Gross, and Krook ([3]).

Equation (2), taking account of (4) and (6), assumes the form

$$(\mathbf{v}\nabla)f = -\varphi/\tau. \tag{7}$$

The properties of the gas near the wall change very strongly only in a thin wall layer (of thickness of order λ); therefore, in the general case one may write:

$$|\partial \ln \varphi/\partial x| \sim 1/\lambda.$$

On the other hand, φ depends on y only implicitly—through the temperature $T(y)$ and the density $n(y)$. Under real conditions both of these quantities change substantially only over distances much greater than λ, i.e.

$$|\partial \ln \varphi/\partial y| \sim |\partial \ln T/\partial y| \ll 1/\lambda.$$

From these relations it follows that

$$|\partial\varphi/\partial y| \ll |\partial\varphi/\partial x|,$$

which makes it possible to regard φ as a function only of x and $\mathbf{v}$. Taking this into account, it is easy to write the equation satisfied by the function φ:

$$v_x \partial\varphi/\partial x + \varphi/\tau = -v_y \partial f_u^{(0)}/\partial y. \tag{8}$$

It is convenient to introduce φ^+ and φ^-, with x-components of the velocity directed away from the wall and toward the wall, respectively. The solution of equation (8) can be represented in the form:

$$\varphi^\pm = -\tau v_y \frac{\partial f_u^{(0)}}{\partial y}\left[1 + C^\pm \exp\left(-\frac{x}{\tau v_x}\right)\right]; \tag{9}$$

$C^- = 0$ from the condition of boundedness of φ^-; C^+ is found from the boundary condition at the wall.

The detailed behavior of molecules striking the wall surface is complicated; therefore we introduce the accommodation coefficient q, indicating

the part of the molecules striking the surface of the wall that come into thermodynamic equilibrium with the wall and leave it with an isotropic Maxwellian velocity distribution. A fraction $(1-q)$ of the molecules is specularly reflected from the surface of the wall.

Introducing the accommodation coefficient q, it is easy to obtain the relation between the complete distribution functions f^+ and f^- at $x=0$,

$$f^+(0,y,\mathbf{v}) = qf_0^{(0)} + (1-q)f^-(0,y,-v_x,v_y,v_z), \tag{10}$$

where

$$f_0^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{m}{2kT}\mathbf{v}^2\right),$$

$$f^+ = f_u^{(0)}(y,\mathbf{v}) + \varphi^+(x,\mathbf{v}), \qquad f^- = f_u^{(0)}(y,\mathbf{v}) + \varphi^-(x,\mathbf{v}).$$

From (10) it is easy to obtain the boundary condition for the correction to the distribution function

$$\varphi^+(0,\mathbf{v}) = q\left(f_0^{(0)} - f_u^{(0)}\right) + (1+q)\varphi^-(0,-v_x,v_y,v_z). \tag{11}$$

Substituting into the boundary condition (11) the expressions for the correction to the distribution function (9), we determine the constant C^+, after which the final expression for the functions $\varphi^\pm$ will have the form

$$\varphi^\pm = -\tau v_y\frac{\partial f_u^{(0)}}{\partial y}\left[1 - q\exp\left(-\frac{x}{\tau v_x}\right)\right] + q\left(f_0^{(0)} - f_u^{(0)}\right)\exp\left(-\frac{x}{\tau v_x}\right),$$

$$\varphi^- = -\tau v_y\frac{\partial f_u^{(0)}}{\partial y}, \tag{12}$$

where

$$\frac{\partial f_u^{(0)}}{\partial y} = f_u^{(0)}\left(-\frac{5}{2T} + \frac{m}{2kT^2}\mathbf{v}^2\right)\frac{dT}{dy}. \tag{13}$$

We determine the thermal slip velocity from the condition of equality of the tangential momentum flux at the wall and at infinity:

$$\int_{-} v_x v_y f^{-}(0, \mathbf{v})\, d\mathbf{v} + \int_{+} v_x v_y f^{+}(0, \mathbf{v})\, d\mathbf{v} = \int v_x v_y f(\infty, \mathbf{v})\, d\mathbf{v}, \tag{14}$$

where the domain of integration is given by the relations

$$\int_{-} d\mathbf{v} = \int_{-\infty}^{0} dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z, \qquad \int_{+} d\mathbf{v} = \int_{0}^{\infty} dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z.$$

The functions φ^{+} and φ^{-} at the surface of the wall have the form

$$\varphi^{+}(0, \mathbf{v}) = -\tau v_y \frac{\partial f_u^{(0)}}{\partial y}(1-q) + q\left(f_0^{(0)} - f_u^{(0)}\right), \qquad \varphi^{-}(0, \mathbf{v}) = -\tau v_y \frac{\partial f_u^{(0)}}{\partial y}. \tag{15}$$

Substituting (15) into (14), we obtain

$$q\int_{+} v_x v_y^2 \tau \frac{\partial f_u^{(0)}}{\partial y}\, d\mathbf{v} + q\int_{+} v_x v_y \left(f_0^{(0)} - f_u^{(0)}\right) d\mathbf{v} = 0. \tag{16}$$

The thermal slip velocity u is small in comparison with the velocities of the thermal motion of the molecules; therefore $f_u^{(0)}$ may be expanded in a Taylor series in powers of

$$\frac{m}{kT} u v_y$$

$$f_u^{(0)} = f_0^{(0)}\left[1 + \frac{m}{kT} u v_y\right]. \tag{17}$$

From (16) and (17) we have

$$u = \tau \frac{kT}{2m} \frac{d \ln T}{dy}. \tag{18}$$

The quantity τ entering this formula is found from the condition that, at large distances from the wall, the distribution function coincide with the Chapman-Enskog solution (4)

$$\tau = \frac{15}{16}\lambda \left(\frac{\pi m}{2kT}\right)^{1/2}.$$

Introducing the kinematic viscosity $\nu = \frac{15}{16}\lambda(2kT\pi/m)^{1/2}$, formula (18) can be written in the form

$$u = \frac{3}{4}\nu\,\frac{d\ln T}{dy}.$$

The results obtained show that, although the Bhatnagar method gives a dependence of the distribution function on the coordinate x, it nevertheless does not change the slip velocity obtained by Maxwell. The reason is that the Bhatnagar method uses a very crude approximation of the collision integral in the Boltzmann kinetic equation. A more rigorous treatment of this integral will be carried out in our next work.

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REFERENCES

[1] J. C. Maxwell, Phil. Trans. Roy. Soc. London, **170**, 231 (1879).
[2] B. V. Deryagin, S. P. Bakanov, DAN, **141**, No. 2, 384 (1961).
[3] P. L. Bhatnagar, E. P. Gross, M. Krook, Phys. Rev., **94**, 511 (1954).
[4] S. Chapman, T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, IL, 1960, p. 150.

Note: Figure translations are in progress. See original paper for figures.