ON REGIONS OF IMPOSSIBILITY OF MOTIONS IN THE THREE-BODY PROBLEM

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ON REGIONS OF IMPOSSIBILITY OF MOTIONS IN THE THREE-BODY PROBLEM

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§ 1. We shall consider the motion of three bodies P_1, P_2, P_3, with masses m_1, m_2, m_3, attracting one another according to Newton' s law, in a barycentric coordinate system $Oxyz$, oriented so that the direction of the axis Oz coincides with the direction of the vector of the angular momentum of the system, which, as is known, is a constant quantity. We shall consider systems in which the modulus of this vector $C \neq 0$. In such systems, by Sundman' s theorem, a triple collision of the bodies is impossible [2].

The energy integral has the form $T = U + h$, where T is the kinetic energy of the system; $U = f(m_1 m_2 r_{12}^{-1} + m_1 m_3 r_{13}^{-1} + m_2 m_3 r_{23}^{-1})$ is the force function (f is the gravitational constant, r_{ij} is the distance between P_i and P_j); h is the energy constant. From the qualitative point of view, the most difficult case (in view of the variety of possible motions) is $h < 0$, for which it is convenient to introduce the constant $h' = -h > 0$. From the integral $T = U - h'$ it is readily obtained that in this case complete disintegration of the system is impossible, since

$$\min_{i \neq j} r_{ij} \leqslant \delta = \frac{f}{h'}(m_1 m_2 + m_1 m_3 + m_2 m_3). \tag{1}$$

In the present paper, for the case $C \neq 0,\ h < 0\ (h' > 0)$, regions of impossibility of motion are found for each of the three bodies, as well as the conditions for their existence and their basic properties.

§ 2. We shall rely on the inequality relating the moment of inertia I_ζ of a system of n bodies with masses m_i relative to a certain axis $O\zeta$

$$\left(I_\zeta = \sum_{i=1}^{n} m_i(\xi_i^2 + \eta_i^2) \text{ in the rectangular coordinate system } O\xi\eta\zeta \right),$$

the kinetic energy T_ζ of the motion of the system in projection onto the plane perpendicular to the axis $O\zeta$

$$\left(T_\zeta = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot\xi_i^2 + \dot\eta_i^2)\right),$$

and the angular momentum L_ζ of the system relative to the axis $O\zeta$

$$\left(L_\zeta = \sum_{i=1}^{n} m_i(\xi_i\dot\eta_i - \eta_i\dot\xi_i)\right),$$

namely

$$I_\zeta T_\zeta \geqslant 1/2 L_\zeta^2. \tag{2}$$

We give a scheme of the proof of (2). Make the change of variables

$$\xi = \xi' \cos\omega t - \eta' \sin\omega t, \quad \eta = \xi' \sin\omega t + \eta' \cos\omega t, \quad \zeta = \zeta',$$

where ω is an arbitrary but constant angular velocity of rotation of the system $O\xi'\eta'\zeta'$ about the axis $O\zeta' = O\zeta$. Express I_ζ, L_ζ, and T_ζ in terms of the new variables. After this it turns out that

$$T_\zeta = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot\xi_{i'}^2 + \dot\eta_{i'}^2) + \omega L_\zeta - \frac{1}{2}\omega^2 I_\zeta \geqslant \omega L_\zeta - \frac{1}{2}\omega^2 I_\zeta.$$

Thus, $I_\xi\omega^2 - 2L_\xi\omega + 2T_\xi \geqslant 0$ for any ω. For $I_\xi = 0$ also $L_\xi = 0$, so that (2) is satisfied. Therefore one may assume that $I_\xi > 0$. From the inequality for a quadratic trinomial it is clear that it cannot have distinct real roots, i.e., its discriminant is nonpositive, which leads to (2).

Since $T \geqslant T_\xi$, $I_\xi T \geqslant \frac{1}{2}L_\xi^2$. Applied to the three-body problem, it follows from this that

$$I_\xi(U - h') \geqslant \frac{1}{2}C^2 \sin^2\varphi, \tag{3}$$

if the axis $O\xi$ makes an angle φ with the plane Oxy. The plane Oxy, by virtue of the choice of the coordinate system, is called the Laplace plane (^{1}).

It is easy to infer from (3) that the location of the three bodies on one straight line and, in particular, a double collision are possible only in the case when this straight line belongs to the Laplace plane.

§ 3. We now turn to the main purpose of the paper. We shall denote by the indices i, j, k the numbers $1, 2, 3$, taken in an arbitrary but fixed order. We shall call the body P_i distant (and the bodies P_j and P_k close) if $r_{ij} \geqslant r_{jk}$, $r_{ik} \geqslant r_{jk}$ (in particular cases two, or each of the three, bodies may turn out to be distant).

In this section we shall find a restriction on the spatial motion of the body P_i under the assumption that it is distant. After this it will be easy to consider the cases when P_i is close to P_j or P_k. The region of impossibility of motion of the body P_i, common to all three cases, will be an unconditional region of impossibility of motion of the body P_i. Since the index i may take any of the values $1, 2, 3$, we shall find these regions for each of the three bodies.

Thus, let P_i be the distant body. Then $r_{jk} \leqslant \delta$, where δ is given by (1). Denote OP_i by r_i, and the angle between OP_i and the plane Oxy by φ_i. We apply (3) to the axis OP_i, assuming that $\varphi_i \neq 0$ (for $\varphi_i = 0$ we do not obtain a restriction on r_i). The center of mass $O_{(jk)}$ of the bodies P_j and P_k lies on the straight line OP_i in the direction opposite to the body P_i, at a distance from it equal to $\rho_i = mr_i/(m_j + m_k)$, where m is the mass of the entire system. The distances of P_j from $O_{(jk)}$ and of P_k from $O_{(jk)}$ are: $r_{j(jk)} = m_k r_{jk}/(m_j+m_k) \leqslant m_k\delta/(m_j+m_k)$, $r_{k(jk)} = m_j r_{jk}/(m_j+m_k) \leqslant m_j\delta/(m_j+m_k)$. Obviously, $r_{ij} \geqslant \rho_i - r_{j(jk)} \geqslant (mr_i - m_k\delta)/(m_j + m_k)$, $r_{ik} \geqslant \rho_i - r_{k(jk)} \geqslant (mr_i - m_j\delta)/(m_j + m_k)$. The right-hand sides of these estimates are positive for $r_i > A_i$, where $A_i = \delta \max(m_j, m_k)/m$. Only these values of r_i will be considered (the opposite inequality $r_i \leqslant A_i$ already constitutes a restriction on r_i). For $\varphi_i \neq 0$ the bodies do not lie on one straight line. In the plane $OP_{iP_{jP}}k$ draw through the point $O_{(jk)}$ a straight line perpendicular to $O_{(jk)}OP_i$. Denote the projection of the segment $P_{jP}k$ onto this straight line by l_{jk}, so that $r_{jk} \geqslant l_{jk} > 0$. The moment of inertia of the system with respect to the axis OP_i is equal to $I_i = m_{jm}kl_{jk}^2/(m_j+m_k)$. For the force function we obtain the estimate $U = f(m_{jm}kr_{jk}^{-1} + m_{im}jr_{ij}^{-1} + m_{im}kr_{ik}^{-1}) < f\{m_{jm}kl_{jk}^{-1} + m_i(m_j + m_k)[m_j/(mr_i - m_k\delta) + m_k/(mr_i - m_j\delta)]\}$. As a result, from (3) we obtain:

$$\frac{m_{jm}k}{m_j + m_k}\left[h' - fm_i(m_j + m_k)\left(\frac{m_j}{mr_i - m_k\delta} + \frac{m_k}{mr_i - m_j\delta}\right)\right] l_{jk}^2 - \frac{fm_j^2 m_k^2}{m_j + m_k} l_{jk} + \frac{C^2}{2}\sin^2\varphi_i < 0. \tag{4}$$

Denote $F_i(r_i) = m_j/(mr_i - m_k\delta) + m_k/(mr_i - m_j\delta)$. As r_i varies from A_i to $+\infty$, F_i decreases monotonically from $+\infty$ to zero. Therefore there is a unique value A_i', $A_i < A_i' < +\infty$, such that $F_i(A_i') = h'/fm_i(m_j + m_k)$. We shall consider only $r_i > A_i'$. Then the previous condition $r_i > A_i$ is automatically satisfied, and the coefficient of l_{jk}^2 in (4) is positive. In view of inequality (4), the quadratic trinomial has two distinct real roots, i.e., a positive discriminant. This leads to the desired inequality $F_i(r_i) > \Phi_i(\sin\varphi_i)$, where $\Phi_i(u) = [h' - f^2 m_j^3 m_k^3/2(m_j + m_k)C^2u^2]/fm_i(m_j + m_k)$. As $|\varphi_i|$ varies from

from 0 to $\pi/2$ $\Phi_i(\sin\varphi_i)$ increases monotonically from $-\infty$ to $\Phi_i(1) < F_i(A_i')$. If $h'C^2 > f^2 m_j^3 m_k^3/2(m_j + m_k)$, then $\Phi_i(\sin\varphi_i)$ passes through zero at

$$|\varphi_i| = \varphi_i^* = \arcsin fm_{jm}k\left[m_{jm}k/2(m_j + m_k)h'C^2\right]^{1/2}.$$

$0 < \varphi_i^* < \pi/2$. It is clear that for $0 \leq |\varphi_i| \leq \varphi_i^*$ there will be no restriction on

r_i. For $\varphi_i^* < |\varphi_i| \leq \pi/2$, on the contrary, an upper restriction on r_i appears: $r_i < R_i(\varphi_i)$ for $r_i > A_i'$, $\varphi_i^* < |\varphi_i| \leq \pi/2$. From the properties of $F_i(r_i)$ and $\Phi_i(\sin\varphi_i)$ it follows that, as $|\varphi_i|$ varies from φ_i^* to $\pi/2$, $R_i(\varphi_i)$ decreases monotonically from $+\infty$ to $R_i(\pi/2)$. Since $F_i(A_i') > \Phi_i(1)$, $R_i(\pi/2) > A_i'$. Consequently, in the result obtained by us

$$r_i < R_i(\varphi_i), \quad \varphi_i^* < |\varphi_i| \leq \pi/2, \tag{5}$$

the condition $r_i > A_i'$ can be removed (from $r_i \leq A_i'$, (5) follows all the more).

It remains to give an explicit expression for (5). From $F_i(r_i) > \Phi_i(\sin\varphi_i)$, under the condition $r_i > A_i$, follows the inequality:

$$r_i^2 - \frac{m_i + m_k}{m}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right] r_i + \frac{\delta}{m^2}\left[m_{jm}k\delta + \frac{m_j^2 + m_k^2}{\Phi_i(\sin\varphi_i)}\right] < 0. \tag{6}$$

The quadratic trinomial (6) has two unequal positive roots: $\widetilde{R}_i(\varphi_i)$ and $R_i(\varphi_i)$, $\widetilde{R}_i < R_i$. The solution of inequality (6) has the form $\widetilde{R}_i < r_i < R_i$. From the properties of $F_i(r_i)$ it follows that always $\widetilde{R}_i < A_i$. Therefore $A_i < r_i < R_i$, since (6) was derived under the assumption $r_i > A_i$. But now it can be removed. Hence (5) is valid, where

$$R_i(\varphi_i) = \frac{m_j + m_k}{2m}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right] +$$

$$+\left\{\frac{(m_j + m_k)^2}{4m^2}\left[\delta + \frac{1}{\Phi_i(\sin\varphi_i)}\right]^2 - \frac{\delta}{m^2}\left[m_{jm}k\delta + \frac{m_j^2 + m_k^2}{\Phi_i(\sin\varphi_i)}\right]\right\}^{1/2}. \tag{7}$$

For the practical construction of the boundary (5), however, it is more convenient to put the inequality $F_i(r_i) > \Phi_i(\sin\varphi_i)$ in the form

$$|\sin\varphi_i| < fm_{jm}k\left\{\frac{m_{jm}k}{2(m_j + m_k)C^2\left[h' - fm_i(m_j + m_k)F_i(r_i)\right]}\right\}^{1/2}. \tag{8}$$

§ 4. Let now P_i be close to P_j, so that $r_{ij} \leq \delta$. The distance of P_i from $O_{(ij)}$ is equal to $r_{i(ij)} = m_{jr_{ij}}/(m_i + m_j) \leq m_j\delta/(m_i + m_j)$. Since P_k is a distant body, $r_k < R_k(\varphi_k)$, $\varphi_k^* < |\varphi_k| \leq \pi/2$. Hence, for $O_{(ij)}$:

$$r_{(ij)} < m_{kR}k|\varphi_{(ij)}|/(m_i + m_j), \quad \varphi_k^* < |\varphi_{ij}| \leq \pi/2.$$

Thus,

$$r_i < R_i^j(\varphi_i), \quad \varphi_k^* < |\varphi_i| \leq \pi/2, \tag{9}$$

where $R_i^j(\varphi_i)$ passes farther than the boundary for $O_{(ij)}$ and parallel to it at a distance $m_j\delta/(m_i + m_j)$. Similarly, if P_i is close to P_k,

$$r_i < R_i^k(\varphi_i), \quad \varphi_j^* < |\varphi_i| \leq \pi/2, \tag{10}$$

where $R_i^k(\varphi_i)$ passes farther than the boundary for $O_{(ij)}$ and parallel to it at a distance $m_k\delta/(m_i+m_k)$.

Choosing, for a given $|\varphi_i|$, the largest of the right-hand sides of (5), (9), (10), one can construct an unconditional boundary of the region of impossibility of motion of the body P_i. It exists for $\varphi^* < |\varphi_i| \leq \pi/2$, where $\varphi^* = \max(\varphi_i^*, \varphi_j^*, \varphi_k^*) = \max(\varphi_1^*, \varphi_2^*, \varphi_3^*)$. If the masses of the bodies are numbered in decreasing order: $m_1 \geq m_2 \geq m_3$, then from the expressions for the angles φ_i^* it follows that

$$\varphi^* = \arcsin f m_1 m_2 \left[m_1 m_2/2(m_1+m_2)h'C^2\right]^{1/2}.$$

Thus, an unconditional boundary of the region of impossibility of motion for each of the three bodies will exist if

$$h'C^2 > f^2 m_1^3 m_2^3/2(m_1+m_2), \tag{11}$$

where m_1, m_2 are the two largest masses. It is easy to construct examples in which condition (11) is indeed satisfied. For lack of space we shall confine ourselves to the simplest example. For Lagrange' s triangular solution with equal masses,

$$\varphi^* = \arcsin \frac{1}{3\sqrt{2}} \simeq 13°38'.$$

By slightly changing the initial conditions, one can obtain from the planar motion a spatial motion for which the angle φ^* will differ little from the value given above.

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REFERENCES

1. G. N. Duboshin, *Celestial Mechanics. Basic Problems and Methods*, Moscow, 1963.

2. J. D. Birkhoff, *Dynamical Systems*, Moscow, 1941.

Note: Figure translations are in progress. See original paper for figures.