The convergence of the method of collocation by lines

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Abstract

We consider the Dirichlet and Neumann boundary value problems for the equations

$$L_u \equiv \Delta u = v(x, y), \tag{1}$$

$$L_u \equiv \Delta u - \lambda u = v(x, y) \tag{2}$$

in the square $R[0 \le x, y \le \pi]$. Approximate solutions are sought in the form

$$u_m^{(1)} = \sum_{k=1}^{m} f_k(x) \sin ky \tag{3}$$

for the Dirichlet problem and in the form

$$u_m^{(1)} = \sum_{k=1}^{m} \varphi_k(x) \cos ky \tag{4}$$

for the Neumann problem. According to the method of collocation along lines, the functions $f_k(x)$ and $\varphi_k(x)$ are determined from a system of ordinary differential equations

$$Lu_m^{(i)}(x, y_j) = v(x, y_j) \quad (i = 1, 2), \tag{5}$$

$$y = y_j \in (0; \pi) \quad (j = 1, 2, \dots, m;\ i = 1), \tag{6}$$

$$y = y_j \in [0; \pi] \quad (j = 0, 1, \dots, m;\ i = 2) \tag{7}$$

with the boundary conditions

$$f_k(0) = f_k(\pi) = 0 \quad (k = 1, 2, \dots, m;\ i = 1), \tag{8}$$

$$\varphi_k'(0) = \varphi_k'(\pi) = 0 \quad (k = 0, 1, \dots, m; i = 2). \tag{9}$$

Under certain requirements imposed on the function $v(x, y)$ and the choice of collocation lines (6) and (7), we prove the solvability of system (5) with conditions (8) or (9). Furthermore, we investigate the rate of convergence of the sequences of approximate solutions $\{(3), (5), (8)\}$ and $\{(4), (5), (9)\}$ to the corresponding exact solutions $u^{(1)}$ and $u^{(2)}$.

Similar results are obtained for the equation

$$\Delta u - \lambda \sum_{k+l=0}^{1} a_{kl} \frac{\partial^{k+l}}{\partial x^k \partial y^l} u = v(x, y), \quad a_{kl} = \text{const}$$

with Dirichlet and Neumann boundary conditions specified on the boundary of the square $R[-\pi \le x, y \le \pi]$, as well as for the equation

$$\Delta u - \lambda w(x, y) u = v(x, y)$$

with the boundary conditions

$$u|_\gamma = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=\pi} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=\pi} = 0.$$

Bibliography: 3 items.

Full Text

Preamble

In this section, we consider the boundary value problems for the equation $\Delta u = v(x, y)$ in the domain $D = \{0 < x, y < \pi\}$. We analyze both the Dirichlet (D) and Neumann (N) conditions on the boundary Γ. Specifically, for the Neumann problem, we assume the consistency condition $\iint_D v(x, y) dxdy = 0$.

Let the approximate solution be represented in the form:

$$u_m^{(1)}(x, y) = \sum_{k=1}^{m} f_k(x) \sin ky \quad \text{(for problem D)}$$

$$u_m^{(2)}(x, y) = \sum_{k=0}^{m} \phi_k(x) \cos ky \quad \text{(for problem N)}$$

The functions $f_k(x)$ and $\phi_k(x)$ are determined by solving the system of ordinary differential equations resulting from the substitution of these forms into the original equation, subject to the boundary conditions $f_k(0) = f_k(\pi) = 0$ and $\phi_k'(0) = \phi_k'(\pi) = 0$.

1. Error Estimates for the Poisson Equation

We define the partial sums of the Fourier series for the source term $v(x, y)$ as $S_m[v(x, y); y]$ for the sine expansion and $C_m[v(x, y); y]$ for the cosine expansion. The approximation error is characterized by the remainders:

$$R_m^{(1)}(v) = \max |v(x, y) - S_m[v(x, y); y]| \to 0 \quad \text{as } m \to \infty$$

$$R_m^{(2)}(v) = \max |v(x, y) - C_m[v(x, y); y]| \to 0 \quad \text{as } m \to \infty$$

The solutions to the boundary value problems can be expressed using the Green' s functions g_D and g_N:

$$u^{(1)}(x, y) = \iint_D g_D(x, y; \xi, \eta) v(\xi, \eta) d\xi d\eta$$

$$u^{(2)}(x, y) = \iint_D g_N(x, y; \xi, \eta) v(\xi, \eta) d\xi d\eta + C_2$$

By comparing the exact solution $u^{(k)}$ with the approximate solution $u_m^{(k)}$, we establish the following convergence rates for the function and its derivatives:

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}[u^{(k)} - u_m^{(k)}] = O[R_m^{(k)}(v)] \quad (k = 1, 2;\ 0 \le i + j \le 1)$$

$$\max |\Delta[u^{(k)} - u_m^{(k)}]| = O[R_m^{(k)}(v)]$$

2. Extension to the Helmholtz Equation

Consider the equation $\Delta u - \lambda u = v(x, y)$ with Dirichlet or Neumann boundary conditions. We assume λ is not an eigenvalue of the corresponding operator. The solution can be represented via an integral equation of the second kind:

$$u(x, y) = \lambda \iint_D g(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta + \iint_D g(x, y; \xi, \eta) v(\xi, \eta) d\xi d\eta$$

For the Neumann problem, the Green' s function g_N is modified to account for the constant term. We define the integral operator A such that the equation becomes $u - \lambda A u = f$. The approximate solution u_m satisfies a corresponding discretized system.

The convergence of the method for the Helmholtz equation is linked to the convergence of the Fourier series of the source term and the kernel. Specifically, if $R_m(k)$ and $R_m(v)$ are the approximation errors for the kernel and the source term respectively, the error in the solution satisfies:

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}[u - u_m] = O[R_m(k) + R_m(v)] \quad (0 \le i + j \le 3)$$

$$\max |\Delta[u - u_m]| = O[R_m(k) + R_m(v)]$$

3. Generalizations and Higher-Order Operators

The method can be extended to higher-order biharmonic-type equations of the form $\Delta^2 u - \lambda u = v(x, y)$ with appropriate boundary conditions on D. Let $u|_{y=0,\pi} = 0$ and $\Delta u|_{y=0,\pi} = 0$. Using a similar expansion in terms of eigenfunctions $\sin ky$, we reduce the problem to a system of fourth-order ordinary differential equations for the coefficients $f_k(x)$.

The error analysis follows the same logic as the second-order case. If the source term $v(x, y)$ belongs to the class $L_2(D)$ and satisfies the necessary boundary conditions, the approximate solution u_m converges to the exact solution u in the corresponding Sobolev space.

References

1. Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis*. 1962.
2. Smirnov, V. I. *A Course of Higher Mathematics*, Vol. IV, 1945.
3. Nikolsky, S. M. *On the solution of boundary value problems by variational methods*. Mat. Sbornik, 1963.

Note: Figure translations are in progress. See original paper for figures.