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## Abstract

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### CYBERNETICS AND CONTROL THEORY

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# CONNECTING DYNAMIC MODELS OF PRODUCTION INTO SCHEMES

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This paper investigates the question of connecting dynamic models of production, possessing individual values of the duration of the production cycle and of the reproduction coefficient, into parallel, sequential, and complex schemes (see ([3])).

The direct connection of productions operating in parallel into schemes, by virtue of the linearity of the model, presents no problem; however, taking into account the multidimensionality of production schemes, which arises mainly because of their great logical breadth, it should be noted that here there arises the problem of aggregating productions operating in parallel into one enlarged production that is, in a certain sense, equivalent to them.

The latter problem, in view of the very great diversity in the functional behavior of productions determined by various characteristics, precludes obtaining an exact solution for it. A satisfactory approximate solution can be obtained if the aggregation is carried out with allowance for the "proximity of productions in their characteristics." On this basis there arises the question of indicators of the quality of production in various modes of operation of production.

The solution of the question of connecting individual productions into sequential and complex schemes is carried out in the paper with allowance for a complicating circumstance permitting return movements of resources (feedbacks) along the line of reproduction.

Let us note that all the investigation considered in the present paper is carried out with allowance for the possibility and in the interests of the convenience of formulating optimality problems both within the framework of dynamic programming (see ([1])) and within the framework of the mathematical theory of optimal processes ([2], [4], [5]).

**Description of the model.** On the basis of ([3]), under conditions of a stable economy, let us consider a model of production with changing capacity. In doing so, we shall take into account the possible delay in the increase of production capacity from the moment the corresponding resources are allocated. Such a model is represented by the scheme of Fig. 1 and is described by the system

$$dx_1/dt = (\beta - 1)u_1 - w_2 - w_3; \tag{1}$$

$$dx_2/dt = aw_2(t - \nu); \tag{2}$$

$$\tau\beta u_1 \leqslant x_2; \tag{3}$$

$$\tau A(\beta)u_1 + \tau B(\beta)w_2 + \tau B(\beta)w_3 \leqslant x_1, \tag{4}$$

in which $x_1(t)$ is the total quantity of circulating ingredients at the disposal of the production; $x_2(t)$ is the productivity (capacity) of production per cycle; $\tau$ is the duration of the complete production cycle; $\beta$ is the reproduction coefficient; $a$ is the coefficient of capacity increase depending on the allocated resources; $\nu$ is the magnitude of the delay in the increase of productivity.

The intensities (rates) of movement of products intended for reproduction, accumulation, and removal (import) are denoted, respectively—

···respectively through $u_1(t), u_2(t), u_3(t)$ $(-w_3(t))$. The expressions $A(\beta), B(\beta)$ have the form $A(\beta) = (\beta - 1)/\ln\beta,\ B(\beta) = (\beta\ln\beta - \beta + 1)/\ln\beta(\beta - 1)$.

By the conditions of a calm economy it is understood that the intensities $w_2(t)$ and $w_3(t)$ are piecewise-analytic, slowly varying functions, allowing one to retain only the principal terms in expressions (1)–(2) and (4).

For $w_2(t), w_3(t) = c\gamma^{t/\tau}$ the estimate given in paper (3) formulates these conditions in the form $0.0019 \sim e^{-2\pi} \lll \gamma \lll e^{2\pi} \sim 530$.

**Aggregation.** Consider $n$ dynamically operating productions working in parallel, each described separately by the system (1)–(4), in which the values of the indicators $x_1, x_2$, the intensities $u_1, w_j$ $(j = 2, 3)$, and the parameters $\alpha, \beta, \tau, \nu$ are marked above by the superscript $i = 1, 2, \dots, n$. We shall regard this collection of productions as one production, calling it a parallel aggregate.

*Fig. 1*

The planning interval $[T_1, T_2]$ for each $i$-th production is divided into sets $\tau_\mu^i$ and $\tau_\rho^i$, pertaining to the operation of this production, respectively, in the capacity regime or in the resources regime, in other words, under the conditions

$$\tau^i \beta^i u_1^i = x_2^i \tag{5}$$

or

$$\tau^i A(\beta^i) u_1^i + \tau^i B(\beta^i) w_2^i + \tau^i B(\beta^i) w_3^i = x_1^i. \tag{6}$$

The set on which both equalities (5) and (6) hold simultaneously is included in $\tau_\mu^i$. By the same symbols $\tau_\mu^i$ and $\tau_\rho^i$ we shall denote the measure of the corresponding sets.

For a separate production we introduce into consideration constants which we shall call elementary quality indicators:

$$k_1 = (\beta - 1)/\tau\beta, \quad k_2 = \ln\beta/\tau, \quad k_3 = \alpha, \quad k_4 = \alpha/C(\beta), \quad k_5 = 1/C(\beta), \tag{7}$$

where $C(\beta) = (\beta - 1)B(\beta)/A(\beta)$, and the averaged and incremented quantities

$$\bar{x}_2 \tau_\mu = \int_{\tau_\mu} x_2(t)\, dt, \qquad \tilde{x}_1 \tau_\rho = \int_{\tau_\rho} x_1(t)\, dt, \qquad \bar{\Delta}(\nu) x_2 = \int_{\tau_\mu} dx_2(t + \nu),$$

$$\hat{\Delta}(\nu) x_2 = \int_{\tau_\rho} dx_2(t + \nu), \tag{8}$$

$$\tilde{w}_3 \tau_\rho = \int_{\tau_\rho} w_3(t)\, dt, \qquad \Delta x_j = x_j(T_2) - x_j(T_1) \quad (j = 1, 2).$$

Assuming that the operation of the parallel aggregate is described by the corresponding indicators $x_1(t), x_2(t)$ and intensities $u_1(t), w_j(t)$ $(j = 2, 3)$, satisfying the capacity constraints (3) and the resource constraints

$$\tau A(\beta) u_1 + \tau_2 B(\beta) w_2 + \tau_3 B(\beta) w_3 \leq x, \tag{9}$$

where $\beta, \tau, \tau_2, \tau_3$ are certain constants, one can, analogously to the preceding, introduce the sets $\tau_\mu$ and $\tau_\rho$, the elementary quality indicators

$$k_1 = (\beta - 1)/\tau\beta, \qquad k_2 = \ln\beta/\tau, \qquad k_3 = \alpha, \qquad k_4 = \alpha\tau/C(\beta)\tau_2,$$

$$k_5 = \tau/C(\beta)\tau_3 \tag{10}$$

Fig. 2

Figure 1: Fig. 2

Fig. 3

Figure 2: Fig. 3

and the averaged and incremented quantities in the form (8).

Let us make the assumptions:

I. The sum of the mean values $x_1^i$ and $x_2^i$ for the individual production units in the capacity and resource regimes, respectively, coincides with the corresponding mean values $x_1$ and $x_2$ for the parallel aggregate.

II. The total costs for accumulation of capacity and for export for the individual production units in the different regimes coincide with the corresponding costs for the aggregate.

Fig. 2

Fig. 3

Fig. 4

**Theorem 1.** *The description of a parallel aggregate of production units possessing identical lag coefficients $\nu^i = \nu$ $(i = 1, \dots, n)$, by the system (1), (2), (3), (9), under assumptions I and II, ensures fulfillment of the relations:*

$$\sum_{i=1}^{n} \Delta x_1^i = \Delta x_1, \qquad \sum_{i=1}^{n} \Delta(\nu) x_2^i = \Delta(\nu) x_2,$$

*if $\alpha, \beta, \tau, \tau_2, \tau_3$ are chosen as solutions of the system of functional equations (10), in which $k_1, k_2, k_3, k_4, k_5$ have the following averaged values:*

$$k_1 = \sum_{i=1}^{n} k_1^i \bar{x}_2^i \tau_\mu^i \Big/ \sum_{i=1}^{n} \bar{x}_2^i \tau_\mu^i, \qquad k_2 = \sum_{i=1}^{n} k_2^i \tilde{x}_1^i \tau_\rho^i \Big/ \sum_{i=1}^{n} \tilde{x}_1^i \tau_\rho^i,$$

$$k_3 = \sum_{i=1}^{n} \bar{\Delta}(\nu) x_2^i \Big/ \sum_{i=1}^{n} \frac{\bar{\Delta}(\nu) x_2^i}{k_3^i}, \qquad k_4 = \sum_{i=1}^{n} \widetilde{\Delta}(\nu) x_2^i \Big/ \sum_{i=1}^{n} \frac{\widetilde{\Delta}(\nu) x_2^i}{k_4^i},$$

$$k_5 = \sum_{i=1}^{n} \widetilde{w}_3^i \tau_\rho^i \Big/ \sum_{i=1}^{n} \frac{\widetilde{w}_3^i \tau_\rho^i}{k_5^i}.$$

Fig. 4

Figure 3: Fig. 4

**Sequential connection of productions.** To compose systems of differential equations and inequalities describing the functioning of productions connected sequentially in such a way that, taken together, they form a ring with intermediate outputs (Fig. 2), we shall use the method of **decomposing a sequential scheme into parallel cycles (parallels)**, respectively attached to the individual productions.

Let $u_j^i$ $(i, j = 1, \dots, n)$ denote the intensities of the input of ingredients into the $i$-th production, belonging to the $j$-th parallel cycle, and let $v^i$ $(i = 1, \dots, n)$ denote the intensities of redistribution of ingredients from the $(i-1)$-st to the $i$-th parallel cycle, measured at the input to the $i$-th production (see Fig. 3, where the $i$-th production with the parallels passing through it is shown).

Taking into account that $u^i = u_1^i + u_2^i + \dots + u_n^i + v^i$ $(i = 1, \dots, n)$ and $u_k^i = u_k^k / \beta_k^i$, where $\beta_k^i = \beta_{k-1} \dots \beta_i$ for $i < k$, $\beta_k^i = 1$ for $i = k$; $\beta_k^i = \beta_n \dots \beta_i \beta_{k-1} \dots \beta_1$ for $i > k \neq 1$; $\beta_k^i = \beta_n \dots \beta_i$ for $k = 1,\ i > 1$, we obtain

$$u_k^i = \frac{\beta}{\beta_k^i(\beta - 1)} \left[ (u^k - v^k) - \frac{1}{\beta_k} (u^{k+1} - v^{k+1}) \right], \tag{11}$$

where $i, k = 1, \dots, n$ modulo $n$, $\beta = \prod_{i=1}^n \beta_i$. Composing, for each parallel cycle, the differential equation and inequalities, with the aid of (11) we obtain the desired system:

$$dx_i/dt = \beta_i u^i - u^{i-1} - w_i, \qquad \tau_i^0 \beta_i u^i \leq x_{n+i},$$

$$\tau E(\beta)(\beta_i u^i - u^{i+1}) + \tau G(\beta)(\beta_i v^i - v^{i+1}) + \tau \overline{B}(\beta) w_i \leq x_i,$$

in which

$$\tau = \sum_{i=1}^n \tau_i, \qquad E(\beta) = A(\beta)/(\beta - 1), \qquad G(\beta) = E(\beta) + B(\beta).$$

**Complex connection of productions.** A connection of dynamic productions in which every production consumes, as raw material, the output of every production of the scheme will be called a **maximally connected connection** (see Fig. 4). If the connection satisfies the latter condition only partially, then, in the general case, we shall call it **complex**. To describe the operation of complexly connected productions, analogously to what was said earlier, we shall apply the method of decomposing the paths of motion of the product in the scheme into **parallels or groups of cycles corresponding to individual productions**. In this case the parallels (groups of cycles) consist of cycles containing productions in different numbers and in different orders. We use the technological relations

$$u_j^i(t) = \alpha_j^i \overline{u}^i(t) \qquad (\alpha_j^i = \text{const},\ \alpha_j^i \geq 0)$$

for the intensities $u_j^i(t)$ of the input of the $j$-th raw material into the $i$-th production.

We introduce: a) the intensities $u_{j,k}^i(t)$ of the input of the $j$-th raw material into the $i$-th production, belonging to the $k$-th cycle; b) the intensities $v_j^i(t)$ of redistribution of the quantities $x_i(t)$ $(i = 1, \dots, n)$. Taking into account that from the intensities $u_{j,k}^i(t)$ $(i, j = 1, \dots, n;\ k = 1, \dots, m$, where $m$ is the total number of cycles) one can compose a basis of $m$ functions, we compose, as before, for each production a differential equation and a capacity constraint, and also, for each cycle belonging to the parallel of the given production, a resource constraint.

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*Note: Figure translations are in progress. See original paper for figures.*