A STUDY OF S. N. BERNSTEIN’ S AVERAGED INTERPOLATION PROCESS IN THE COMPLEX DOMAIN

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Abstract

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MATHEMATICS

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A STUDY OF S. N. BERNSTEIN'S AVERAGED INTERPOLATION PROCESS IN THE COMPLEX DOMAIN

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1°. As is known, there exists no system of interpolation nodes for which, for every function continuous on $[-1, 1]$, the Lagrange interpolation process converges uniformly. In connection with this, S. N. Bernstein (^{1}) introduced sequences of polynomials of the form

$$M_n(x, f) = \sum_{k=1}^{n} m_{n,k}(f)\, l_{n,k}(x), \tag{1}$$

where $l_{n,k}(x)$ are the fundamental Lagrange polynomials for a given system of nodes $x_k^{(n)}$, $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$, and $m_{n,k}(f)$ are certain linear functionals. S. N. Bernstein considered the following two processes for the P. L. Chebyshev nodes

$$x_k^{(n)} = \cos \frac{2k-1}{2n} \pi :$$

$$1) \quad m_{n,k}(f) = \frac{1}{4} f(x_{k-1}^{(n)}) + \frac{1}{2} f(x_k^{(n)}) + \frac{1}{4} f(x_{k+1}^{(n)}), \quad k = 1, 2, \ldots, n; \tag{2}$$

2) let p be a given natural number and

$$m_{n,k}(f) = f(x_k^{(n)}) \quad \text{for } k \not\equiv 0 \pmod{2p},$$

$$m_{n,k}(f) = \sum_{j=1}^{p} f(x_{2p(t-1)+2j-1}) - \sum_{j=1}^{p-1} f(x_{2p(t-1)+2j}) \tag{3}$$

$$\text{for } k = 2pt, \quad t = 1, 2, \ldots, [n/2p].$$

These processes, as S. N. Bernstein showed (¹), converge uniformly on $[-1,1]$ for any function $f(x)$ continuous on $[-1,1]$. D. L. Berman ($^{2-4}$) studied the processes (1), defined by means of (2) and (3) for other nodes, and established the rate of convergence for an arbitrary function continuous on $[-1,1]$. All these studies concerned the real domain.

The passage from the real domain to the complex domain is, as a rule, associated with certain difficulties. The first work in which such a passage was carried out was the work of L. V. Kantorovich (^{5}), in which the classical S. N. Bernstein polynomials are studied in the complex domain. Until recently the question of convergence of the processes (1) in the complex domain remained unstudied. In 1965 D. L. Berman (^{6}) showed that the process (1), (3) (for $p=1$) converges in the complex domain and proposed generalizing his results. The present note is devoted to a generalization of the results of D. L. Berman (^{6}).

2°. Let B be a bounded domain of the z-plane whose boundary L is a closed analytic Jordan curve; let B' be the complement of B in the extended plane; let $w=\varphi(z)$ be the function mapping B' conformally onto the domain $|w|>1$ and normalized by the conditions $\varphi(\infty)=\infty$, $\varphi'(\infty)=c>0$; and let $z=\psi(w)$ be the inverse function to $w=\varphi(z)$.

Set $\theta_k^{(n)}=2\pi k/n,\ k=0,\pm1,\pm2,\dots;\ n=1,2,\dots;\ w_k^{(n)}=e^{i\theta_k^{(n)}}$;

$$z_k^{(n)}=\psi(w_k^{(n)});\quad \omega_n(z)=\prod_{k=1}^{n}(z-z_k^{(n)}),\quad l_{n,k}(z)=\omega_n(z)/(z-z_k^{(n)})\,\omega_n'(z_k^{(n)}).$$

Let n and s be natural numbers; $p_{\nu,k}^{(n)}$, $\nu=0,\pm1,\dots,\pm s$, $k=0,\pm1,\pm2,\dots$, be real numbers such that $p_{\nu,k}^{(n)}\equiv p_{\nu,k+n}^{(n)}$.

To every function $f(z)$ regular in B and continuous on $\overline{B}$, we assign the polynomial

$$M_n(z,f)=\sum_{k=1}^{n}f(z_k^{(n)})l_{n,k}^*(z),\qquad l_{n,k}^*(z)=\sum_{\nu=-s}^{s}p_{\nu,k}^{(n)}l_{n,k+\nu}(z).\tag{4}$$

It is easy to see that

$$M_n(z,f)=\sum_{k=1}^{n}m_{n,k}(f)l_{n,k}(z),\qquad m_{n,k}(f)=\sum_{\nu=-s}^{s}p_{\nu,k-\nu}^{(n)}f(z_{k-\nu}^{(n)}),$$

and, consequently, the polynomials (1) specified by means of the numbers (2) and (3) are special cases of the polynomials (4).

For the study of convergence of the polynomials (4), it is convenient to represent them in the form

$$M_n(z,f) = I_{1,n}(z,f) + I_{2,n}(z,f),$$

$$I_{1,n}(z,f) = \sum_{k=1}^{n} p_k^{(n)} f(z_k^{(n)}) l_{n,k}(z), \qquad p_k^{(n)} = \sum_{\nu=-s}^{s} p_{\nu,k}^{(n)}, \tag{5}$$

$$I_{2,n}(z,f) = \sum_{k=1}^{n} f(z_k^{(n)}) \sum_{\nu=-s}^{s} p_{\nu,k}^{(n)} \left[l_{n,k+\nu}(z) - l_{n,k}(z)\right].$$

We shall need several lemmas below.

Lemma 1. The formula

$$l_{n,k}(z) = \frac{1+\varepsilon_k^{(n)}(z)}{2\pi i} \, \frac{d\psi(e^{i\theta_k^{(n)}})/d\theta}{z_k^{(n)} - z} \, \frac{2\pi}{n},$$

holds, where $\varepsilon_k^{(n)}(z) \to 0$ as $n \to \infty$ uniformly in z inside B and uniformly in k.

Proof. We have

$$l_{n,k}(z) = \frac{\omega_n(z)}{(z - z_k^{(n)})\omega_n'(z_k^{(n)})} = \frac{\omega_n(z)}{in(-c^n)(z_k^{(n)} - z)} \cdot \frac{d\psi(e^{i\theta_k^{(n)}})/d\theta}{[\omega_n'(z_k^{(n)})/inc^n]\,[d\psi(e^{i\theta_k^{(n)}})/d\theta]},$$

and the assertion of the lemma follows from the following facts ([7],[8]):

1) $\omega_n(z)/(c^n) \to 1$ as $n \to \infty$ uniformly inside B;

2) $[\omega_n'(z_k^{(n)})/inc^n]\,[d\psi(e^{i\theta_k^{(n)}})/d\theta] \to 1$ as $n \to \infty$ uniformly in k.

Lemma 2. Let s be fixed; suppose the sequence

$$p_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{\nu=-s}^{s} |p_{\nu,k}^{(n)}|, \quad n = 1, 2, \dots,$$

is bounded. Then $I_{2,n}(z) \to 0$ as $n \to \infty$ uniformly inside B.

Proof. Introduce the notation: E is an arbitrary closed set of points belonging to B; δ is the distance from E to L;

$$M = \max_{z \in L} |f(z)|, \quad \varepsilon_n = \sup_{k,\, z \in E} |\varepsilon_k^{(n)}(z)|,$$

$$m' = \max_{-\pi \le \theta \le \pi} \left| d\psi(e^{i\theta})/d\theta \right|, \qquad m'' = \max_{-\pi \le \theta \le \pi} \left| d^2\psi(e^{i\theta})/d\theta^2 \right|.$$

Relying on Lemma 1, for $z \in E$ we obtain

$$|l_{n,k+\nu}(z) - l_{n,k}(z)| \leq$$

$$\leq \frac{2\varepsilon_n m'}{\delta} \frac{1}{n} + \frac{1}{n} \left| \frac{d\psi(w_{k+\nu}^{(n)})/d\theta}{z_{k+\nu}^{(n)} - z} - \frac{d\psi(w_k^{(n)})/d\theta}{z_k^{(n)} - z} \right| \leq$$

$$\leq \frac{1}{n} \left[\frac{2\varepsilon_n m'}{\delta} + \frac{m''}{\delta} \frac{2\pi|\nu|}{n} + \frac{{m'}^2}{\delta^2} \frac{2\pi|\nu|}{n} \right],$$

and therefore, if $p_n \leq m, \;\; n = 1, 2, \ldots$, we have

$$|I_{2,n}(z, f)| \leq Mm \frac{2}{\delta} \left[\varepsilon_n m' + \left(m'' + \frac{{m'}^2}{\delta} \right) \frac{\pi s}{n} \right].$$

Remark. More delicate arguments make it possible to prove Lemma 2 under weaker restrictions on s and on the numbers $p_{\nu,k}^{(n)}$.

Lemma 3. Let q be a fixed natural number, and let the numbers $p_k^{(n)}$ have the following properties:
1) $p_k^{(n)} \geq 0, \quad k = 0, \pm 1, \pm 2, \ldots;$

$$2)\;\; p_{k+q}^{(n)} = p_k^{(n)}, \quad k = 0, \pm 1, \pm 2, \ldots; \; 3) \sum_{k=1}^{n} p_k^{(n)} = n.$$

Then $I_{1,n}(z, f) \to f(z)$ as $n \to \infty$, uniformly inside B.

The proof is based on Lemma 1.

Lemma 4. Under the conditions of Lemma 2, the polynomials $M_n(z, f)$ are uniformly bounded inside B.

Now the following is obvious.

Theorem 1. Suppose that the conditions of Lemmas 2 and 3 are satisfied. Then, for every function $f(z)$ regular in B and continuous in $\overline{B}$, the sequence of polynomials $M_n(z, f)$, $n = 1, 2, \ldots$, converges uniformly inside B to $f(z)$.

3°. On the boundary L of the domain B, the polynomials (4) may diverge.

Theorem 2. Let B be the disk $|z| < 1$; then there exist numbers $p_{\nu,k}^{(n)}$ and a function $f(z)$, regular in B and continuous in $\overline{B}$, such that at some point z_0 on the circle $|z| = 1$ the equality

$$\lim_{n \to \infty} M_n(z_0, f) = \infty$$

will hold.

Proof. Let s, r, q be fixed natural numbers, with q an odd number; let the sequence

$$\sum_{\nu=-s}^{s} |p_{\nu,k}^{(n)}|, \quad n = 1, 2, \dots,$$

be bounded, let $p_k^{(n)} = q$ for $k = tq + r$, $t = 0, \pm 1, \pm 2, \dots$, and let $p_k^{(n)} = 0$ for the remaining k. It is not difficult to show that in this case the sum $I_{2,nq}(z, f)$ is uniformly bounded on $\overline{B}$. Further, one verifies that $I_{1,nq}(z_0, f)$, $z_0 = e^{(q+2r)\pi i/nq}$, is the value of the Lagrange interpolation polynomial at the nodes $\zeta_k^{(n)} = 2e^{(2k-1)\pi i/n}$ for the function $f(z_0\zeta)$ at the point $\zeta = 1$, and, by Fejér' s theorem (9), there exists a function $f(z)$, regular in B and continuous in $\overline{B}$, such that

$$\overline{\lim_{n\to\infty}} I_{1,nq}(z_0, f) = \infty.$$

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Note: Figure translations are in progress. See original paper for figures.