ON A MATHEMATICAL MODEL OF PRODUCTION

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Abstract

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ON A MATHEMATICAL MODEL OF PRODUCTION

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1°. Many mathematical models of production can be described in terms of mappings that assign to a point of some finite-dimensional space a bounded subset of, generally speaking, another finite-dimensional space.

In this note a general model of this type is investigated. The properties possessed by this model are studied under the assumption that the mappings describing it satisfy certain restrictions. In particular, the question of the asymptotic behavior of optimal trajectories is considered.

For an exact description of the model we introduce some notation. Let $n_1 \le n_2 \le \cdots \le n_t \le \cdots$ be a sequence of natural numbers. Consider a system of finite-dimensional normed spaces $m_1 \subset m_2 \subset \cdots \subset m_t \subset \cdots$. Here m_t is an n_t-dimensional space; if $x = (x^1, \dots, x^{n_t}) \in m_t$, then $\|x\| = \max_{i \le n_t} |x^i|$. We shall regard the space m_t as a subspace of $m_{t'}$ $(t' > t)$, spanned by the first n_t unit vectors. By $P_{t,t'}$ we denote the projection from $m_{t'}$ into m_t. We shall assume that the spaces m_t are partially ordered in the natural way. By K_t we denote the cone of positive elements of the space m_t, and by Ξ_t the totality of all bounded subsets of K_t.

The model under consideration is specified by: a) a sequence of natural numbers $n_1 \le n_2 \le \cdots \le n_t \le \cdots$; b) a system of mappings a_t acting from K_t into Ξ_{t+1}.

Economically, t may be interpreted as the number of a time period, and the numbers $1, 2, \dots, n_t$ as the numbers of products available at the beginning of period t. Applying the mapping a_t to the vector $x = (x^1, \dots, x^{n_t}) \in K_t$ means that the products with numbers $1, 2, \dots, n_t$, expended at the beginning of period t in amounts $x_1, x_2, \dots, x^{n_t}$, respectively, can be processed by the end of this period into products with numbers $1, 2, \dots, n_{t+1}$; the set $a_t(x)$ consists of all output vectors $y = (y^1, y^2, \dots, y^{n_{t+1}})$ that can be obtained with expenditures x in period t. The mappings a_t may also have a somewhat different meaning, namely, if some part z_t of the produced product must be used in period t for consumption, then the set $a_t(x)$ consists of all vectors $y - z_t$, where y is an output vector that can be obtained with expenditures x in period t $(y \ge z_t)$.

2°. In investigating the model described, meaningful results can be obtained under the assumption that certain restrictions are imposed on the mappings a_t. Let us list the main such restrictions.

1) **Monotonicity:** if $x', x'' \in K_t$, $x' \geq x''$, then $a_t(x') \supset a_t(x'')$.
2) **Upper semicontinuity:** if $x_n \in K_t$, $x_n \to x$, $y_n \in a_t(x_n)$, $y_n \to y$, then $y \in a_t(x_n)$.
3) **Concavity:** if $x', x'' \in K_t$, $\alpha \in [0, 1]$, then $a_t(\alpha x' + (1-\alpha)x'') \supset \alpha a_t(x') + (1-\alpha)a_t(x'')$.
4) **Positive homogeneity:** if $x \in K_t$, $\lambda \geq 0$, then $a_t(\lambda x) = \lambda a_t(x)$.
5) The mappings a_t are bounded in the aggregate: for any positive-

for every c there is a number M_c, independent of t, such that if $x \in K_t$ $(t = 1, 2, \ldots)$, $\|x\| \leq c$, then $\sup_{y \in a_t(x)} \|y\| \leq M_c$.

b) The sequence a_t is monotone: if $t' > t$, then for $x \in m_{t'}$,

$$a_{t'}(x) \supset a_t(P_{t,t'}x).$$

Let $\xi \in \Xi_\tau$. Consider the sets $A_\tau^t(\xi)$ $(\tau = 1, 2, \ldots;\ t = 0, 1, 2, \ldots)$, defined as follows: $A_\tau^0(\xi) = \xi$;

$$A_\tau^t(\xi) = \bigcup_{x \in A_\tau^{t-1}(\xi)} a_{\tau+t-1}(x) \qquad (t = 1, 2, \ldots).$$

Theorem 1. *From the monotonicity of the mappings a_t follows the boundedness of $A_\tau^t(\xi)$; from the upper semicontinuity of a_t and the closedness of ξ follows the closedness of $A_\tau^t(\xi)$; from the concavity of a_t and the convexity of ξ follows the convexity of $A_\tau^t(\xi)$.*

Consequence. *Since $A_\tau^1(\{x\}) = a_\tau(x)$, the assertion of Theorem 1 is also valid for the sets $a_\tau(x)$.*

In what follows, instead of $A_1^t(\xi)$ we shall write $A^t(\xi)$; instead of $A^t(\{x\})$ we shall write $A^t(x)$.

Let us note that the scheme described includes, in particular, the Gale model ([1]) and a model of the Gale type ([2]). Let us also note the closeness of the model considered to the model described by Yu. N. Tyurin ([3]).

3°. Put $K_t^* = \{f \in m_t^* \mid f \geq 0\}$. A functional $f \in K_t^*$ will be called *prices in period t*. If $f \in K_t^*$, $\xi \in \Xi_t$, then put $f(\xi) = \sup_{x \in \xi} f(x)$. We shall assume that the mappings a_t are monotone and upper semicontinuous.

Let $f_T \in K_T^*$. By an (f_T, x)-*optimal trajectory* we shall mean a finite sequence $\{x_t\}_{t=1}^T$ such that $x_1 = x$, $x_t \in a_{t-1}(x_{t-1})$ $(t = 2, 3, \ldots, T)$. Let $\xi \in \Xi_1$, ξ be closed. Consider the triple $(\bar{\chi}, \bar{\varphi}, \alpha)$. Here

$$\bar{\chi} = \{\bar{x}_t\}_{t=1}^\infty \quad (\bar{x}_1 \in \xi,\ \bar{x}_t \in a_{t-1}(\bar{x}_{t-1}),\ t = 2, 3, \ldots);$$

$$\bar{\varphi} = \{\bar{f}_t\}_{t=1}^{\infty} \quad (\bar{f}_t \in K_t^*,\ \|\bar{f}_t\| = 1); \qquad \alpha = \{\alpha_t\}_{t=1}^{\infty} \quad (0 < \alpha_t < \infty).$$

The triple $(\bar{\chi}, \bar{\varphi}, \alpha)$ will be called an *equilibrium system* on ξ if the following conditions are fulfilled: 1) $\bar{f}_1(\bar{x}_1) > 0$; 2) $\bar{f}_{t+1}(a_t(x)) \leq \alpha_t \bar{f}_t(x)$ $(x \in A^{t-1}(\xi))$; 3) $\bar{f}_{t+1}(\bar{x}_{t+1}) = \alpha_t \bar{f}_t(\bar{x}_t)$ $(t = 1, 2, ...)$.

Put

$$Z_t(\xi) = \{z = (x, y) \in K_t \times K_{t+1} \mid x \in A^{t-1}(\xi),\ y \in a_t(x)\};$$
$$W_\varepsilon^t = \{z = (x, y) \in Z_t(\xi) \mid \alpha_t(1-\varepsilon)\bar{f}_t(x) \geq \bar{f}_{t+1}(y)\}; \qquad W_\varepsilon = \bigcup_t W_\varepsilon^t.$$

On the set $m_t \times m_{t+1}$ introduce a norm by putting, for $z = (x, y) \in m_t \times m_{t+1}$, $\|z\| = \|x\| + \|y\|$. The normed space thus obtained will be denoted by $(m_t \times m_{t+1})_1$. Consider the linear functional $\bar{g}_t \in (m_t \times m_{t+1})_1^*$, defined by the formula $\bar{g}_t = (\alpha_t \bar{f}_t, -\bar{f}_{t+1})$, and let $H_{\bar{g}t}$ be the hyperplane of the functional $\bar{g}_t$.

The introduction of the concept of an equilibrium system is justified by the following theorem.

Theorem 2. *Let the mappings a_t be monotone and upper semicontinuous* $(t = 1, 2, ...)$. *Consider a closed set $\xi \in \Xi_1$ such that there exists an equilibrium system $(\bar{\chi}, \bar{\varphi}, \alpha)$ on ξ. Let $x \in \xi$ be such that for some τ, $\bar{x}_\tau \in A^{\tau-1}(x)$. Let further $k_1 \geq k_2$ be arbitrary positive numbers, $T \geq \tau$, and let the functional $f_T \in K_T^*$ have the following properties: a) $f_T(A^{T-1}(x)) > 0$; b) $k_2 \bar{f}_T \leq f_T \leq k_1 \bar{f}_T$.*

Then, for any $\varepsilon \in (0, 1)$ and any (f_T, x)-optimal trajectory $\{x_t\}_{t=1}^T$, the number of pairs $(x_t, x_{t+1}) \in W_\varepsilon$ does not exceed the number

$$N = \ln\left[\frac{k_2 \bar{f}_1(\bar{x}_1)}{k_1 \bar{f}_1(x)}\right] \Big/ \ln(1-\varepsilon).$$

If, in addition,

$$\inf_t \sup_{z \in Z_t(\xi)} \frac{\bar{g}_t(z)}{\|\bar{g}_t\|\|z\|} > 0 \quad \text{and} \quad 0 < \varepsilon < \inf_t \sup_{z \in Z_t(\xi)} \frac{\bar{g}_t(z)}{\|\bar{g}_t\|\|z\|},$$

the number of pairs (x_t, x_{t+1}) such that

$$\rho\left(\frac{(x_t, x_{t+1})}{\|(x_t, x_{t+1})\|},\ H\bar{g}_t\right) > \varepsilon$$

also does not exceed the number N.

Theorem 2 describes the asymptotic behavior of optimal trajectories. It may be regarded as a generalization of the turnpike theorems in weak form that hold in Gale' s model (see, for example, ([2])).

4°. Let the mappings a_t $(t = 1, 2, ...)$ be monotone and upper semicontinuous, and let $\xi \subset E_1$ be a closed set. An (∞, ξ)-optimal trajectory is a sequence

$\{x_t\}_{t=1}^{\infty}$ such that $x_1 \in \xi$, $x_{t+1} \in a_t(x_t)$, and x_t is a maximal element of the set $A^{t-1}(\xi)$ $(t = 1, 2, \ldots)$.

Theorem 3. Let the mappings a_t $(t = 1, 2, \ldots)$ be monotone and upper semicontinuous. Whatever closed set $\xi \subset E_1$ is given, there exists an (∞, ξ)-optimal trajectory.

Theorem 4. Let the mappings a_t $(t = 1, 2, \ldots)$ be monotone, upper semicontinuous, concave, positively homogeneous, and such that the sets

$$\bigcup_{x \in K_t} a_t(x)$$

are solid. Let the closed convex set $\xi \subset E_1$ either contain an interior point of the cone K_1, or consist of a single point and, in addition, possess the property that the sets $A^t(\xi)$ contain an interior point of the cone K_{t+1} $(t = 1, 2, \ldots)$.

Then, whatever the (∞, ξ)-optimal trajectory $\{\bar{x}_t\}_{t=1}^{\infty}$, there exists an equilibrium system $(\bar{\chi}, \bar{\varphi}, \alpha)$ on ξ such that $\bar{\chi}$ coincides with $\{\bar{x}_t\}_{t=1}^{\infty}$.

5°. A sequence $\{x_t\}_{t=1}^{\infty}$ will be called **consistent** if $x_t \in K_t$, $\sup_t \|x_t\| < \infty$, and for any t and $t' > t$, $P_{t,t'} x_{t'} = x_t$.

Theorem 5. Let the mappings a_t $(t = 1, 2, \ldots)$ be monotone. In order that these mappings be bounded in the aggregate, it is necessary and sufficient that, for every consistent sequence $\{x_t\}_{t=1}^{\infty}$,

$$\sup_t \sup_{y \in a_t(x_t)} \|y\| < \infty.$$

Let $k \le n_t$ and $\varepsilon > 0$. Put

$$V_{k,t,\varepsilon} = \{x = (x^1, \ldots, x^{n_t}) \in m_t \mid \sup_{i \le k} |x^i| \le \varepsilon\}.$$

Theorem 6. Let the mappings a_t $(t = 1, 2, \ldots)$ be monotone and bounded in the aggregate; let the sequence a_t be monotone. Then for every consistent sequence $\{x_t\}_{t=1}^{\infty}$, for every number k and every $\varepsilon > 0$, there is a τ such that for $t \ge \tau$

$$P_{\tau+1,t+1}(a_t(x_t)) \subset a_\tau(x_\tau) + X_{k,\tau+1,\varepsilon}.$$

The economic meaning of Theorem 6 is as follows: in all time periods $t \ge \tau$, the quantity of the i-th product $(i = 1, \ldots, k)$ obtained by some method from the vector x_t will differ from the quantity of this product that can be obtained from the vector x_τ in period τ by no more than ε.

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Note: Figure translations are in progress. See original paper for figures.