THE FOUR-FERMION THIRRING MODEL AND PERTURBATION THEORY

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Abstract

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THE FOUR-FERMION THIRRING MODEL AND PERTURBATION THEORY

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Usually, in various field-theoretical models, an exact solution is obtained at the cost of abandoning certain basic requirements whose fulfillment is indispensable in any realistic theory. In this sense the Thirring model ([1]) is essentially not a model, for the only restriction consists in considering a two-dimensional relativistically invariant theory of a fermion field of zero mass. This theory is specified by the Lagrangian density:

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_{int}(x),$$

$$\mathcal{L}_0(x) = \frac{i}{2}\sum_{\mu=0}^{1} : \left\{ \overline{\psi}\gamma^\mu \frac{\partial\psi}{\partial x_\mu} - \frac{\partial\overline{\psi}}{\partial x_\mu}\gamma^\mu\psi \right\} :, \tag{1}$$

$$\mathcal{L}_{int} = -\frac{g}{2} : \left(\overline{\psi}(x)\psi(x)\right)^2 : .$$

In (1), γ^μ are two-dimensional Dirac matrices, for which one may choose the following representation:

$$\gamma^{(0)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \gamma^{(1)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The free fields are given by the formulas

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int d^2k\, e^{i(kx)} \begin{pmatrix} \psi_1(k^0k) \\ \psi_2(k^0k) \end{pmatrix},$$

$$\psi_1(k^0k) = \delta(k^0 - k)\,[\theta(-k)a(-k) + \theta(k)b^+(k)], \tag{2}$$

$$\psi_2(k^0k) = \delta(k^0 + k)\,[\theta(k)a(-k) + \theta(-k)b^+(k)].$$

The idea of Glaser–Berezhin ([1,2]) was first to solve the problem in the unphysical space $\mathcal{H}_\psi$, in which ψ_i^+ and ψ_i act as creation and annihilation operators, and then to obtain the solution in the physical space $\mathcal{H}_a$ by what is essentially a relabeling of the normal-product symbol.

Let us note that consideration of the problem in the space $\mathcal{H}_\psi$ means that, in applying Wick' s theorem, we use contractions different from the contractions in the physical space $\mathcal{H}_a$. Namely, if in $\mathcal{H}_a$ ($|0\rangle$ is the vacuum in $\mathcal{H}_a$),

$$\underbrace{\psi_1(x)\psi_1^+(y)} = \underbrace{\psi_1^+(x)\psi_1(y)} = \langle 0|\psi_1(x)\psi_1^+(y)|0\rangle = \langle 0|\psi_1^+(x)\psi_1(y)|0\rangle = \delta_+(v(x)-v(y)), \tag{3}$$

$$\underbrace{\psi_2(x)\psi_2^+(y)} = \underbrace{\psi_2^+(x)\psi_2(y)} = \langle 0|\psi_2(x)\psi_2^+(y)|0\rangle = \langle 0|\psi_2^+(x)\psi_2(y)|0\rangle = \delta_-(u(x)-u(y))$$

$$\left(\text{here } \ \delta_\pm(x) = \frac{1}{2\pi}\int_0^\infty d\alpha\, e^{\pm i\alpha x}, \quad v(x) = x - x^0; \quad u(x) = x + x^0\right),$$

then in $\mathcal{H}_\psi$ ($|0\rangle$ is the vacuum in $\mathcal{H}_\psi$)

$$\begin{aligned}
&\underbrace{\psi_1(x)\psi_1^+(y)} = (0|\psi_1(x)\psi_1^+(y)|0) = \delta(v(x)-v(y)),\\
&\underbrace{\psi_1^+(x)\psi_1(y)} = (0|\psi_1^+(x)\psi_1(y)|0) = 0,\\
&\underbrace{\psi_2(x)\psi_2^+(y)} = (0|\psi_2(x)\psi_2^+(y)|0) = \delta(u(x)-u(y)),\\
&\underbrace{\psi_2^+(x)\psi_2(y)} = (0|\psi_2^+(x)\psi_2(y)|0) = 0.
\end{aligned} \tag{4}$$

Let us consider several consequences following from this distinction. We compute the commutators $[\rho_j(x);\rho_j(y)]$, where $\rho_j(x) = \psi_j^+(x)\psi_j(x)$. First we carry out the calculations in $\mathcal{H}_a$:

$$\begin{aligned}
&[\rho_1(x);\rho_1(y)]_- = \delta_+^2(v(x)-v(y)) - \delta_+^2(v(y)-v(x)),\\
&[\rho_2(x);\rho_2(y)]_- = \delta_-^2(u(x)-u(y)) - \delta_-^2(u(y)-u(x)).
\end{aligned} \tag{5}$$

The expressions on the right-hand side of (5) can easily be given a meaning. Using the Fourier transform, we find

$$\delta_+^2(v(x)-v(y)) - \delta_+^2(v(y)-v(x)) = \frac{i}{4\pi}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x}\right)\delta(v(x)-v(y)),$$

$$\delta_-^2(u(x)-u(y)) - \delta_-^2(u(y)-u(x)) = \frac{i}{4\pi}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x}\right)\delta(u(x)-u(y)). \tag{6}$$

If we were to calculate these commutators in $\mathcal{H}_\psi$, they would turn out to be zero, for their nontriviality in $\mathcal{H}_a$ comes from C-number terms arising when the expressions $\rho_j(x)\rho_j(y)$ are expanded in normal products. These terms contain "counter" contractions, which, by virtue of (4), must give zero.

If we pass to the noncovariant form of notation, then *

$$\mathcal{L}_{int}(x) = g : \rho_1(x)\rho_2(x) : . \tag{7}$$

It follows at once from this that in $\mathcal{H}_\psi$

$$S = T\left(\exp i \int \mathcal{L}_{int}\, dx\right) = \exp ig \int : \rho_1(x)\rho_2(x) : \, dx, \tag{8}$$

i.e., we have the Glaser–Berezin result for the S-matrix.

If the Heisenberg field and current are defined by the formulas

$$\Psi(x) = \overset{+}{S}\, T(\psi(x)S); \qquad I(x) = g\, \overset{+}{S}\, T(: (\bar{\psi}(x)\psi(x))\psi(x) : S), \tag{9}$$

then, passing to the noncovariant form of notation and using the commutativity of $\rho_j(x)$ in $\mathcal{H}_\psi$ and formula (8), it is easy to obtain equations of the Glaser type, since

$$I_1(x) = ig\, \overset{+}{S}\, T(: \psi_1^+(x)\psi_1(x)\psi_2(x) : S) = ig\rho_1(x)\Psi_2(x),$$

$$I_2(x) = -ig\, \overset{+}{S}\, T(: \psi_2^+(x)\psi_2(x)\psi_1(x) : S) = -ig\rho_2(x)\Psi_1(x). \tag{10}$$

It is not hard to understand that carrying relations (8), (10) over into $\mathcal{H}_a$ by redefining the symbol of the normal product gives rise to serious doubts. This is especially clear in the example of the current. No relations of the type (10) can be obtained in $\mathcal{H}_a$, which is a direct consequence of the noncommutativity of the operators ρ_i in $\mathcal{H}_a$ (3).

* The use of the same symbols for the normal and chronological products in $\mathcal{H}_a$ and $\mathcal{H}_\psi$ should not lead to misunderstanding, since each time we invest them with a quite definite meaning.

The definition of the Heisenberg field in $\mathcal{H}_a$ reduces to the very complicated problem of defining the S-matrix off the mass shell while satisfying the basic requirements of covariance, unitarity, and causality. But even on the mass surface, a fully correct argument in favor of the validity of the Glaser-Berezinskii prescription would so far be only the still unproved coincidence of the formal expansion of the scattering matrix proclaimed in $\mathcal{H}_a$ with the perturbation-theory series. We now turn to a partial solution of this problem.

Let us consider the vertex function on the mass surface. In what follows it is convenient to assume that the interaction is included between two types of fields, ψ_1 and ψ_2. A second-order perturbation-theory calculation shows that, on the mass shell, two diagrams with external ends of the same type make a zero contribution, while the sum of the two possible diagrams with different ends is associated with the coefficient function

$$A^{(2)}(p_1p_2p_3p_4) = -\frac{ig^2}{4\pi}\ln\left(\frac{s+i\varepsilon}{u+i\varepsilon}\right)\delta(p_1+p_2-p_3-p_4), \tag{11}$$

where $s = (p_1+p_2)^2,\ u = (p_1-p_4)^2$. But on the mass surface $s = -u$. It follows from this that, in second-order perturbation theory,

$$A^{(2)} = -\frac{g^2}{4}\delta(p_1+p_2-p_3-p_4)\varepsilon(s),$$

which exactly coincides with the second order of perturbation theory for the vertex in $\mathcal{H}_\psi$. Such a coincidence permits one to suggest that the two-dimensionality of the problem and the zero mass are strong conditions, sufficient to eliminate the difference between the theories in $\mathcal{H}_a$ and $\mathcal{H}_\psi$ on the mass shell. The example of the second order shows that one may try to narrow the class of diagrams contributing to the vertex function on the mass surface. It turns out that, in any order of perturbation theory, diagrams with ends of the same type on the mass surface give zero contribution.

We pass to the proof. The contribution of a diagram with four external ends of N-th order to the coefficient function (we omit the δ-function giving the energy-momentum conservation law) in the α-representation is determined by the formula

$$\frac{(ig)^N}{N!}\frac{(-1)^{l+N-1}}{(16\pi)^{N-1}}\lim_{r\to 0}\int\frac{\prod\limits_{\text{over all lines}}\left\{P\left(\frac{\partial}{\partial r}\right)d\alpha\right\}}{D(\alpha)}\times\exp\left[i\frac{A(\alpha,pp)-2B(\alpha,rp)-K(\alpha,rr)}{D(\alpha)}\right]. \tag{12}$$

To each internal line there correspond a Feynman parameter α and a vector r, and to each external line its own momentum p; l is the total number of closed loops of types 1 and 2; $P(\partial/\partial r) = \partial/\partial r^0 - \partial/\partial r$, if the line is of type 1, and $P(\partial/\partial r) = \partial/\partial r^0 + \partial/\partial r$, if it is of type 2. The prescription for constructing the forms $D(\alpha)$ and the form $A(\alpha, pp)$, bilinear in p, is presented in work [4]. The forms B and K, essential for what follows, were obtained by B. M. Stepanov* and are constructed in the following way. The form B is linear in all r_i, and the coefficient of r_i is obtained as follows. All trees of the diagram that contain the i-th line are taken (a tree is a maximally weakly connected diagram containing all vertices of the original one). For each of them one forms the product of the parameters α not belonging to the tree under consideration. This product is multiplied by the sum of the momenta entering one of the halves of the tree, this sum being taken with the plus sign if the i-th line enters the half of the tree under consideration, and with the minus sign if it leaves it. After this, the sum over all such trees of the resulting expressions is taken. The form K is quadratic in r and is the sum over all cycles of products of squares

* Unpublished.

the sums of all r (the sum is taken with the direction of r taken into account) belonging to each cycle, by the product of the parameters α that do not belong to the tree with the given cycle. (A tree with the given cycle is a configuration that becomes a tree when the lines of the given cycle are removed.) Consider an N-th order diagram with identical ends, say of type 1. It is clear that there will be two more internal lines of type 2 than of type 1. Let us perform in (12) first all differentiations corresponding to lines of type 2. After the first differentiation a factor of the form

$$\sum [P(\alpha)L_1(p) + Q(\alpha)L_1(r)]$$

will appear before the exponential, where $P(\alpha)$, $Q(\alpha)$ are certain functions of α; $L_1(x)$ is a linear function depending on x only through the combination $x^0 - x$. Hence it is clear that, since for external ends of type 1 on the mass shell, by virtue of (2), $p^0 = p$, this factor reduces to $\sum Q(\alpha)L_1(r)$. Under subsequent differentiations this pre-exponential factor is not differentiated, since $(\partial/\partial r^0 + \partial/\partial r)L_1(r) = 0$. As a result of all differentiations with respect to the lines of type 2, we obtain a pre-exponential factor whose effective degree in r will be N. The remaining differentiations, corresponding to lines of type 1, are, as already stated, $N-2$. Therefore in the limit $r_i \to 0$ we obtain zero. An analogous proof can be carried out for diagrams with external ends of type 2. Unfortunately, this result alone is still insufficient for reducing the entire perturbation-theory series to the expression following from the Glaser–Berezin result. However, apparently one can show that on the mass surface all vertex diagrams either reduce to constants or are functions of the sign s, i.e., in this case the perturbation-theory series has the structure of the Glaser–Berezin solution. The agreement of perturbation theory in second order on the mass surface with this solution allows one to hope not only for qualitative, but also for quantitative agreement.

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CITED LITERATURE

1. V. Glaser, *Nuovo Cim.*, **9**, 990 (1958).

2. F. A. Berezin, *The Method of Second Quantization*, Moscow, 1965.

3. A. S. Wightman, Lectures at the French Summer School of Theoretical Physics, Cargèse, Corsica, 1964.

4. O. I. Zav' yalov, *ZhETF*, **47**, 1100 (1964).

Note: Figure translations are in progress. See original paper for figures.