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Abstract

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ON A PROBLEM OF G. SZEGŐ, M. KAC, G. BAXTER, AND J. HIRSCHMAN

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1. Let

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

$\{c_{-n}\bar{c}_n\}_0^\infty$ be a nonnegative function of class L_1 on $[-\pi, \pi]$; introduce for consideration the "parameters"

$$a_n = (-1)^n \frac{|c_{i-k+1}|_0^n}{D_n(f)}, \qquad D_n(f) = |c_{i-k}|_0^n > 0, \qquad |a_n| < 1 \quad (n = 0, 1, \ldots). \tag{1}$$

G. Szegő ([2]) indicated conditions sufficient for the existence of the limit

$$\lim_{n\to\infty} \{D_n(f)[G(f)]^{-n-1}\} = \exp\left\{\sum_{n=1}^{\infty} n|d_n|^2\right\}, \qquad \ln f(\theta) \sim \sum_{n=-\infty}^{\infty} d_n e^{in\theta}, \tag{2}$$

where $G(f) = \exp(d_0)$ is the geometric mean of the function $f(\theta)$ on the interval $[-\pi, \pi]$.

In the present note we find conditions sufficient for (2) to hold and more general than the conditions of the authors named in the title of the article.

2. When any one of the mutually equivalent conditions

$$\ln f(\theta) \in L_1(-\pi, \pi), \qquad G(f) > 0, \qquad \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

is satisfied, one can construct a function

$$\pi(z)=\sum_{n=0}^{\infty}\gamma_n z^n,$$

which is regular in the domain $|z|<1$, is not equal to zero, and $1/\pi(z)\in H_2$, and moreover almost everywhere on the interval $[-\pi,\pi]$ we have

$$f(\theta)=|\pi_0(\theta)|^{-2},\qquad \pi_0(\theta)=\lim_{r\to 1-0}\pi(re^{i\theta}). \tag{3}$$

Theorem. Let $0<m\le f(\theta)\le M$ for $-\pi\le\theta\le\pi$; then:
1) the existence of the limit in the left-hand side of (2) is equivalent to the convergence of any one of the series

$$\sum_{n=1}^{\infty}n|d_n|^2,\qquad \sum_{n=1}^{\infty}n|c_n|^2,\qquad \sum_{n=1}^{\infty}n|\gamma_n|^2,\qquad \sum_{n=1}^{\infty}n|a_n|^2,$$

$$\sum_{n=1}^{\infty}\left[\omega_2\left(\frac{1}{n};f\right)\right]^2,\qquad \sum_{n=1}^{\infty}\left[\omega_2\left(\frac{1}{n};\pi_0\right)\right]^2 \tag{4}$$

(where $\omega_2(\delta;\varphi)$ is the integral modulus of continuity of the function $\varphi(\theta)\in L_2(-\pi,\pi)$ in the metric of this space); 2) when this condition is satisfied, for the validity of (2) it is sufficient that

$$\lim_{n\to\infty}\varphi_n^*(e^{i\theta})=\pi_0(\theta);\qquad \varphi_n^*(z)=z^n\overline{\varphi_n}(1/z)\quad (n=0,1,...), \tag{5}$$

where the polynomials $\{\varphi_n(z)=a_nz^n+...\}_0^\infty$ are orthonormal on the circle $|z|=1$ with respect to the weight $f(\theta)$, and the convergence must be uniform on the whole interval $[-\pi,\pi]$.

Combining the convergence condition for one of the series (4) with a condition sufficient for (5) to hold, we obtain a number of conditions sufficient for (2) to hold; some of them are given in Table 1.

Table 1

In all conditions 6)–16) we have $0<m\leqslant f(\theta)\leqslant M$, $-\pi\leqslant\theta\leqslant\pi$. By C, V we denote classes of functions that are, respectively, continuous or of bounded variation on the interval $[-\pi,\pi]$. By $(C,1)$ is denoted the class of series summable by the $(C,1)$ method.

No.	Condition	No.	Condition
1	$\sum_{n=0}^{\infty} \lvert a_n \rvert \in (C,1), \quad \sum_{n=1}^{\infty} n \lvert a_n \rvert^2 < \infty$	9	$\sum_{n=1}^{\infty} n \lvert c_n \rvert < \infty$
2	$\sum_{n=0}^{\infty} \lvert a_n \rvert \in (C,1), \quad \lvert a_n \rvert \leqslant \frac{A}{n}$	10	$\sum_{n=0}^{\infty} \lvert c_n \rvert < \infty, \quad \sum_{n=1}^{\infty} n \lvert c_n \rvert^2 < \infty \ln f(\theta) \in C$
3	$\sum_{n=0}^{\infty} \lvert a_n \rvert < \infty, \quad \lvert a_n \rvert \downarrow 0$	11	$\sum_{n=1}^{\infty} \sqrt{n} \lvert c_n \rvert < \infty, \quad f(\theta) \in C,$
4	$\sum_{n=1}^{\infty} \sqrt{\frac{\lvert a_n \rvert^2 + \lvert a_{n+1} \rvert^2 + \cdots}{n}} < \infty$	12	$\sum_{n=0}^{\infty} \lvert c_n \rvert < \infty, \quad f(\theta) \in C, \alpha_n^2 \geqslant \frac{1}{2}(\alpha_{n+1}^2 + \alpha_{n-1}^2)$
5	$\sum_{n=1}^{\infty} n \lvert a_n \rvert^2 < \infty, \quad \ln f(\theta) \in C$	13	$\sum_{n=0}^{\infty} c_n \in (C,1), \quad f(\theta) \in V f(-\theta) = f(\theta), \quad c_n \geqslant 0,$
6	$f'(\theta) \in \operatorname{Lip} \alpha, \quad 0 < \alpha \leqslant 1$	14	$\sum_{n=1}^{\infty} n c_n^2 < \infty$
7	$f(\theta) \in C, \quad x^{-1} \omega(x; f) \in L_1; \sum_{n=1}^{\infty} \omega_2^2 \left(\frac{1}{n}; f\right) < \infty$ or $x^{-1} \omega_2(x; f) \in L_2$	15	$f(\theta) \in V, C; \sum_{n=1}^{\infty} \frac{1}{n} \sqrt{\omega\left(\frac{1}{n}; f\right)} < \infty$ or $x^{-1} \sqrt{\omega(x; f)} \in L_1$
8	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2\left(\frac{1}{n}; f\right) < \infty$ or $x^{-3/2} \omega_2(x; f) \in L_1$	16	Conditions 1)–4) with the replacement of the parameters $\{a_n\}_0^{\infty}$ by the moments $\{c_n\}_0^{\infty}$

3. Conditions 1—5 of the table are imposed on the parameters, with condition 5 belonging to G. Baxter ([4]); conditions 6—8 are imposed on structural characteristics of the function $f(\theta)$, with condition 6 belonging to G. Szegő ([2]); conditions 9—16 are imposed on the function $f(\theta)$ and on its complex Fourier coefficients, with condition 9 belonging to M. Kac ([3]), conditions 11 and 12 to G. Baxter ([4]), and condition 10 to J. Hirschman ([5]).

We note that condition 10 was found by J. Hirschman for the general case of a complex-valued function $f(\theta)$; conditions 5 and 12 were indicated by G. Baxter

as sufficient for the existence of the limit in the left-hand side of (2); from our theorem it follows first of all that these conditions are not only sufficient, but also necessary for the existence of this limit; moreover, it is not difficult to see that condition 12 is equivalent to condition 3, and therefore it is also sufficient for this limit to have the value indicated in (2).

4. In conclusion we make the following remark: if the parameters are subject to the sole condition $\{|a_n|\}_0^\infty < 1$, then the trigonometric moment problem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\sigma(\theta) = c_n \qquad (n = 0, 1, ...), \tag{6}$$

where the moments $\{c_n\}_0^\infty$ are determined from the parameters $\{a_n\}_0^\infty$ by formulas (1), has a unique solution $\sigma(\theta)$; moreover, on the interval $[-\pi, \pi]$ the function $\sigma(\theta)$ is bounded, nondecreasing, and has an infinite set of points of increase.

Under the additional condition $\sum_{n=0}^\infty |a_n| < \infty$ we have $d\sigma(\theta) = f(\theta)d\theta$, where on the interval $[-\pi, \pi]$ the function $f(\theta)$ is continuous and positive and $(5)^*$ holds; therefore, in conditions 1-4 imposed on the parameters, one need not require boundedness of the function $f(\theta)$ from above and below.

Let us also note that condition (5) of the theorem may be replaced by either of the following two less restrictive conditions:

A. Uniformly on $[-\pi, \pi]$ the asymptotic formula holds

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}\sum_{k=0}^{n} a_k \varphi_k^*(e^{i\theta})\right\} = a\pi_0(\theta). \tag{7}$$

B. The function $\ln f(\theta)$ and its conjugate function have no discontinuities of the second kind.

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* See (1), Theorem 8.5.

Note: Figure translations are in progress. See original paper for figures.