ON THE SOLUTION OF EQUATIONS OF THE CHU-LOW EQUATION TYPE

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Abstract

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ON THE SOLUTION OF EQUATIONS OF THE CHU-LOW EQUATION TYPE

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It is known that pion scattering by a fixed source is described in the two-particle approximation under the condition of unitarity by the Chu–Low equation ([1])

$$h_i(\omega) = \frac{\lambda_i}{\omega} + \frac{1}{\pi}\int_1^\infty \left(\frac{\operatorname{Im} h_i(\omega')}{\omega' - \omega} + \frac{A_{ij} \operatorname{Im} h_j(\omega')}{\omega' + \omega}\right) d\omega', \tag{1}$$

where the functions $h_j(\omega)$ are related to the matrix elements of the S-matrix $S_j(\omega)$ by means of the Fourier transform of the source function $u(q^2)$ through the relation

$$S_j(\omega) = 1 + 2iq^{2l+1}u^2(q^2)h_j(\omega); \tag{2}$$

λ_i are given numbers such that $A_{ij}\lambda_j = -\lambda_i$. Equations of type (1) are obtained in the static limit from the dispersion relations rigorously proved by N. N. Bogolyubov for a fixed value of the momentum transfer ([2]). With this approach, instead of (1) one may consider an equation with an arbitrary number of subtractions. The crossing-symmetry matrix A_{ij} is determined by the group with respect to which the interaction of the source and the particle is assumed to be invariant. It is therefore useful first to consider the most general properties of the solutions of (1). We shall formulate them in terms of analytic functions $h_j(z)$ of the complex variable z, where $h_j(\omega) = \lim_{\varepsilon\to+0} h_j(z + i\varepsilon)$. Among such properties of $h_j(z)$ we include the following: meromorphy of $h_j(z)$ in the z-plane with cuts $(-\infty, -1]$, $[+1, +\infty)$, reality of $h_j(z)$, i.e. $h_j^*(z) = h_j(z^*)$, unitarity of $h_j(z)$, crossing symmetry $h_i(-z) = A_{ij}h_j(z)$. The crossing-symmetry condition establishes relations between the symmetric and antisymmetric parts of the functions $h_j(z)$. The unitarity condition leads to the fact that the Riemann surface of the functions $h_j(z)$ is, generally speaking, infinitely sheeted. Indeed, by the change of variable z it is easy to linearize the unitarity condition $z = \sin \pi w$; $H_j(w) = h_j[z(w)]$:

$$\Phi(w) = 2iq^{2l+1}u^2(q^2)\big|_{z(w)}; \qquad H_j^{-1}(w) + \Phi(w) = H_j^{-1}(1-w). \tag{3}$$

A solution of (3) that does not possess symmetry with respect to $z(w)$ contains the new variable w explicitly. Taking this into account, we shall regard $S_i(z)$ as functions of w. Transferring to $S_i(w)$ the properties of the functions $S_i(z)$, we obtain:

$$\begin{aligned} &1) \quad S_i(w) \text{ are meromorphic functions of } w; \qquad 2) \quad S_i^*(w) = S_i(w^*); \\ &3) \quad S_i(w)S_i(1-w) = 1; \qquad 4) \quad S_i(-w) = A_{ij}S_j(w). \end{aligned} \tag{4}$$

The conditions (4) do not determine $S_i(w)$ uniquely. If $S_i(w)$ are any functions satisfying (4), then $S_i[w + i\beta(w)]D(w)$ also satisfy (4), provided that

$$D(w)D(1-w) = 1; \qquad D(w) = D(-w); \qquad D^*(w) = D(w^*);$$

$$\beta(w) = \beta(w+1); \qquad \beta(w) = -\beta(-w); \qquad \beta^*(w) = \beta(w^*). \tag{5}$$

The solution for a two-row matrix of crossing symmetry $A_{l,\,1/2}$, describing the scattering of a particle on a source with moments l, $1/2$, respectively, can be found in ([3]). With the aid of the product (5) it is represented in the form

$$A_{l,\,1/2} = \frac{1}{2l+1}\begin{Vmatrix} -1 & 2(l+1) \\ 2l & +1 \end{Vmatrix}; \qquad S_{l-1/2}(w) = \frac{w-(l+1)}{w+l}\,\varphi_l(w), \tag{6}$$

$$S_{l+1/2}(w) = 1 - \varphi_l(w); \qquad \varphi_l(w) = -\operatorname{tg}\frac{\pi}{2}w\,\frac{\Gamma[(2-w-l)/2]\Gamma[(w-l)/2]}{\Gamma[(1+w-l)/2]\Gamma[(1-w-l)/2]}.$$

For integer l, the function $\varphi_l(w)$ contains a finite number of factors of the form $\pi(w,a) = [w-(1-a)]/(w-a)$, each of which satisfies conditions 1)–3). The solution (6) has the remarkable property, namely

$$\lim_{w\to\infty} S_{l-1/2}(w)/S_{l+1/2}(w) = 1.$$

Below a method will be given for constructing functions $S_i(w)$ in the class of functions for which

$$\lim S_i(w)/S_j(w) = 1.$$

In this case, for each $S_i(w)$ the representation

$$S_i(w) = \Pi_i(w)\varphi(w), \qquad \text{where } \Pi_i(w) = \prod_{a_j=1}^{N_i} \pi(w, a_j). \tag{7}$$

Substituting the representation (7) into the crossing-symmetry condition 4), we obtain that $\Pi_i(w)$ satisfy the equation

$$\Pi_i(-w)\Pi_\lambda(w) = A_{ij}\Pi_j(w), \qquad \text{where } \Pi_\lambda(w) = \prod_{\lambda=1}^{N_\lambda} \frac{w+\lambda}{w-\lambda} \text{ and } N_\lambda \text{ is finite.} \tag{8}$$

For equations (8) there exists an R such that, for $|w| > R$, the expansions

$$\Pi_i(w) = 1 + \sum_{n\geq 0} \frac{a_n^{(i)}}{w^{n+1}}; \qquad \Pi_\lambda(w) = 1 + \sum_{n\geq 0} \frac{\lambda_n}{w^{n+1}}. \tag{9}$$

Substituting the expansions (8) into the unitarity condition 3), one can obtain equations for the coefficients $a_n^{(i)}$

$$a_{2n+1}^{(i)} + \sum_{q=0}^{2n+1} \sum_{q'\leq 0}^{q} (-1)^{q'+1} c_q^{q'} a_{2n-q}^{(i)} a_{q'}^{(i)} = 0, \qquad n \geq 0, \tag{10}$$

where $a_{2n}^{(i)}$ are arbitrary. In an analogous way, from equations (8) there follows a system of crossing-symmetry equations for the coefficients $a_n^{(i)}$

$$A_{ij}a_n^{(j)} = \lambda_n + \sum_{m=1}^{n} \lambda_{n-m} a_{m-1}^{(i)} (-1)^m + (-1)^{n+1} a_n^{(i)}. \tag{11}$$

The different real (by virtue of condition 2)) solutions of the infinite system of equations (10), (11) determine the different solutions of problem (4). The solution of the infinite system of equations (10), (11) is found in a finite number of steps. Indeed, by virtue of the boundedness of the number of poles in $\Pi_i(w)$, and hence also in $\Pi_\lambda(w)$, not all coefficients of the expansions (9) are linearly independent. Only the first N_i coefficients will be linearly independent, where N_i is the number of poles of $\Pi_i(w)$, counted with their multiplicity ([4]). The construction of the functions $S_i(w)$ is completed by finding $\varphi(w)$ from the equations

$$\varphi(w)\varphi(1-w) = 1; \qquad \varphi(-w)/\varphi(w) = \Pi_\lambda(w), \tag{12}$$

a particular solution of which has the form

$$\varphi(w) = \prod_{\lambda} \varphi_\lambda(w),$$

where $\varphi_\lambda(w)$ is defined in (6). We shall illustrate the construction scheme described above by the example of the matrix $A_{l,1}$

$$A_{l,1} = E - \frac{2}{1 + a_1 b_1 - b_2(1 + a_1)} \begin{Vmatrix} 1 & a_1 & -(1 + a_1) \\ b_1 & b_1 a_1 & -b_1(1 + a_1) \\ b_2 & b_2 a_1 & -b_2(1 + a_1) \end{Vmatrix}, \tag{13}$$

$$a_1 = \frac{2l + 1}{2l - 1} \frac{1}{l + 1}; \qquad b_1 = \frac{1}{l + 1}; \qquad b_2 = -\frac{l}{l + 1}.$$

Including $\Pi_1(w)$ in $\varphi(w)$, we obtain from (10), (11) for $S_i(w)$

$$S_0(w) = \frac{w - (2l + 1)/2}{w + (2l - 1)/2} \varphi(w); \quad S_1(w) = \varphi(w); \quad S_2(w) = \frac{w + (2l + 1)/2}{w - (2l + 3)/2} \varphi(w), \tag{14}$$

and in the equation for $\varphi(w)$ we have

$$\Pi_\lambda(w) = \frac{w + (2l + 3)/2}{w - (2l + 3)/2} \frac{w - (2l - 1)/2}{w + (2l - 1)/2}.$$

Finally, the function $\varphi(w)$ is equal to

$$\varphi(w) = \varphi_{(2l+3)/2}(w) \varphi_{-(2l-1)/2}(w) = \frac{w - (2l + 3)/2}{w + (2l + 1)/2} \frac{w + (2l - 1)/2}{w - (2l + 1)/2}. \tag{15}$$

In the course of obtaining (14), (15), the integrality of l was nowhere used; therefore the result is valid for arbitrary real l. Further, using the arbitrariness in (6), one can construct functions satisfying particular equations of type (1).

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Note: Figure translations are in progress. See original paper for figures.