ON THE MOTION OF AN IDEAL FLUID IN A FIELD OF CORIOLIS FORCES

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Abstract

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ON THE MOTION OF AN IDEAL FLUID IN A FIELD OF CORIOLIS FORCES

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1. Formulation of the problem. In large through-flow reservoirs outside the coastal zone, the forces of inertia, as a rule, considerably prevail over the forces of viscosity. This provides grounds for applying the theory of motion of an ideal fluid to the main part of the reservoir.

The steady flow of an ideal incompressible fluid in a field of Coriolis forces is described by the system of equations

$$\operatorname{grad} H = \mathbf{v} \times \operatorname{rot} \mathbf{v} + 2\mathbf{v} \times \vec{\omega}, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is the velocity vector; $\vec{\omega}$ is the vector of the angular velocity of the Earth; $H = p/\rho + v^2/2$; p is pressure; ρ is the density of the fluid. It is easy to see that (1) admits the Bernoulli integral $H = \text{const}$ along a streamline. We shall consider a comparatively narrow class of flows for which all particles of the fluid entering the reservoir have one and the same value of H. Then throughout the entire region (we shall call it the through-flow region) occupied by the particles mentioned, there will be $H = \text{const}$. In this case we have

$$\mathbf{v} \times (\operatorname{rot} \mathbf{v} + 2\vec{\omega}) = 0.$$

Hence three possibilities arise:

$$v = 0, \tag{2}$$

$$\operatorname{rot} \mathbf{v} = -2\vec{\omega}, \tag{3}$$

$$\mathbf{v} = \lambda(\operatorname{rot} \mathbf{v} + 2\vec{\omega}). \tag{4}$$

In the plane case, which is the only one considered in the present paper, possibility (4) is excluded.

In the case of potential motion, the through-flow zone occupies the entire reservoir. For flows of the given class this need not be so.

Indeed, let us introduce the stream function ψ and suppose that on the boundary of the reservoir Γ, $\psi \geqslant 0$, with the value $\psi = 0$ being attained. Then, assuming the given reservoir to be through-flow, we obtain the problem

$$\Delta\psi = 2\omega, \qquad \psi|_{\Gamma} = \varphi(s) \geqslant 0. \tag{5}$$

The solution of (5) has the form

$$\psi = \psi_0 - \frac{\omega}{\pi} \iint_D G(x, y; \xi, \eta)\, d\xi\, d\eta, \tag{6}$$

where ψ_0 is a harmonic function satisfying the condition $\psi_0|_{\Gamma} = \varphi(s)$; G is the Green' s function of the Laplace operator for the region D occupied by the reservoir. Obviously, for a through-flow zone under the given conditions it must be that $\psi \geqslant 0$. However, as follows from (6), if

$$\omega > \min_D \left[\pi\psi_0 \Big/ \iint_D G\, d\xi\, d\eta\right] = K \tag{7}$$

in the domain D there will be points where $\psi < 0$. Consequently, under condition (7) the whole basin cannot be flowing. Thus, cases (2), (3), generally speaking, must be considered together, distinguishing two regions: the flowing one, where relation (3) is satisfied, and the non-flowing one, where the fluid is at rest. It is obvious that for sufficiently small values of the parameter ω, namely, for

$$\omega < K,$$

the non-flowing zone is absent. This case is trivial. As ω increases, or as $\max\psi_0$ decreases, the non-flowing zone increases and in the limit occupies the whole basin. In the opposite case, as $\omega \to 0$, the motion becomes potential.

The problem consists in finding the shape and dimensions of the non-flowing zone. Assuming that the velocity field is continuous, this problem can be formulated as follows:

Find in the domain D a continuously differentiable function $\psi(x, y)$ satisfying the equation

$$\Delta\psi = \begin{cases} 2\omega, & \psi > 0, \\ 0, & \psi \leqslant 0, \end{cases} \tag{8}$$

and the boundary condition

$$\psi|_{\Gamma} = \varphi(s) \geqslant 0. \tag{9}$$

Let us note that if in (8) the inequality signs are replaced by the opposite ones, then one obtains the problem considered in (2), on discontinuous flows according to the scheme of M. A. Lavrent' ev ([3]).

2. Existence theorem. Consider the auxiliary problem:

$$\Delta u = 2\omega\ \mathrm{th}(u/\lambda), \qquad u|_{\Gamma} = \varphi(s). \tag{10}$$

For each $\lambda > 0$ the solution of the problem exists, is unique, and is analytic in the domain D ([1]). Let us show that $u \geqslant 0$. If at some point $u < 0$, then there will be a subdomain $D' \subset D$, on whose boundary $u = 0$, and inside it $u < 0$. Then, according to (10), u is superharmonic in D'. By the minimum principle $u \equiv 0$ in D'; then, by analyticity, $u \equiv 0$ in D, which is impossible.

Consider a sequence $\lambda_n \to 0$. Let u_n be the solution of problem (10) corresponding to $\lambda = \lambda_n$. The function u_n satisfies the integral equation

$$u_n = \psi_0 - \frac{\omega}{\pi} \iint\limits_D \mathrm{th}\,\frac{u_n}{\lambda_n}\, G(x, y; \xi, \eta)\, d\xi\, d\eta. \tag{11}$$

Proceeding from the properties of integrals of potential type and representation (11), we conclude that the sequence $\{u_n\}$ is compact in the space $C^1(D)$. Let a subsequence u_{n_k} converge to a continuously differentiable function u_0. By condition (7), the function u_0 cannot be everywhere positive. We shall show that the open set B of points where $u_0 > 0$ is a domain, if the set α of points of the contour Γ at which $\varphi(s) > 0$ is connected. By the continuity of u_0 up to the boundary, in a neighborhood of the arc α there is a domain $\beta \subset B$. Suppose that the set B contains a point a which cannot be connected with points of the domain β. Take the component $M \subset B$ containing the point a. If at some point m of the boundary of M one has $u_0 > 0$, then $m \in \Gamma$, for otherwise $m \in M$. But then the point a can be connected with the domain β through a neighborhood of the point m. Consequently, on the boundary of M, $u_0 = 0$. Being the limit of a uniformly convergent sequence of subharmonic functions u_{n_k}, the function u_0 is subharmonic in the domain M. By the maximum principle $u_0 \equiv 0$ in M, which contradicts the definition of M.

We shall prove that the limiting function u_0 is a generalized solution of problem (8)–(9). To this end, consider in the domain B the sequence of problems

$$\Delta v_{n_k} = 2\omega\,\mathrm{th}(u_{n_k}/\lambda_k), \qquad v_{n_k}\big|_{\Gamma_B} = u_{n_k}\big|_{\Gamma_B}.$$

It is obvious that in the domain B, $v_{n_k} = u_{n_k}$, i.e. v_{n_k} form a sequence converging in the domain B to the positive function u_0. But in any interior subdomain B' of the domain B

$$\mathrm{th}(u_{n_k}/\lambda_k) \to 1 \quad \text{as } \lambda_k \to 0$$

uniformly; therefore in B, $\Delta u_0 = 2\omega$, so that the function u_0 satisfies equation (8).

Since all the functions u_{n_k} satisfy condition (9), and the sequence converges uniformly, u_0 also satisfies condition (9). By what was proved above, under condition (7), $u_0 = 0$ in $D \setminus B$. The solution of problem (8)–(9) has been constructed.

3. Uniqueness theorem. Suppose problem (8)–(9) has two solutions

$$\Delta\psi_1 = \begin{cases} 2\omega, & (x,y) \in B_1, \\ 0, & (x,y) \in E_1 = D \setminus B_1; \end{cases} \tag{12}$$

$$\Delta\psi_2 = \begin{cases} 2\omega, & (x,y) \in B_2, \\ 0, & (x,y) \in E_2 = D \setminus B_2. \end{cases} \tag{13}$$

Form the difference

$$u = \psi_1 - \psi_2.$$

Two cases are possible: either neither of the sets E_1, E_2 is contained in the other, or one is contained in the other.

Consider, for example, the first case. On the set $E_1 \setminus E_2$, $u < 0$, while on the set $E_2 \setminus E_1$, $u > 0$. Moreover, $u|_\Gamma = 0$. Therefore the domain D is divided into two, D^- and D^+, where respectively $u < 0$ and $u > 0$, and on the boundary of D^- and D^+, $u = 0$. We have $E_1 \setminus E_2 \subset D^-$, $E_2 \setminus E_1 \subset D^+$. According to (12), (13),

$$\Delta u = \begin{cases} -2\omega, & (x,y) \in E_1 \setminus E_2, \\ 0, & (x,y) \in D \setminus E_1 \cup E_2, \\ 2\omega, & (x,y) \in E_2 \setminus E_1; \end{cases}$$

therefore, for example, in D^+ the function u is subharmonic; consequently, inside D^+, $u < 0$, which contradicts the definition of the domain D^+.

The second case is considered analogously.

4. Examples.

1) Consider a one-dimensional steady flow in a rectilinear channel of width H with mean velocity v_0. Problem (8)–(9) takes the form:

$$\psi'' = \begin{cases} 2\omega, & \psi > 0, \\ 0, & \psi \leq 0; \end{cases} \qquad \psi(0) = 0; \qquad \psi(H) = v_0 H.$$

The solution is the expression

$$\psi = \begin{cases} 0, & x \leq a, \\ \omega(x-a)^2, & x \geq a, \end{cases}$$

where $a = H(1 - \sqrt{v_0/\omega H})$ is the width of the stagnant zone. For $v_0/\omega H > 1$ the stagnant zone is absent; it appears when $v_0/\omega H < 1$, i.e. in sufficiently wide channels with slow fluid flow. For example, when $\omega = 2\pi/24$ hour^{-1} and $v_0 = 1$ m/sec, a stagnant zone arises in channels of width greater than 13.7 km.

2) Let us consider a plane flow in a rectangle under the boundary condition

$$\psi_k = M_k s + N_k,$$

where k is the number of the boundary segment ($k = 1, 2, 3, 4$); $s = 1, 2, \ldots, 129$; M_k and N_k are coefficients, the numerical values of which are given in Table 1. This problem roughly models the conditions characteristic of Lake Baikal.

Table 1

k	M_k, m^2/sec	N_k, m^2/sec	s	k	M_k, m^2/sec	N_k, m^2/sec	s
1	6.4	77	$1, 2, \ldots, 19$	4	2.47	952	$63, \ldots, 65$
2	4.56	601	$20, 21, \ldots, 39$	$1'$	-6.4	77	$1, 2, \ldots, 12$
3	5.65	625	$40, 41, \ldots, 62$	$2'$	-2.47	1273	$13, 14, \ldots, 65$

The numerical results obtained by Yu. A. Sozinov by the matrix-sweep method give the picture of the flow shown in Fig. 1. In the central part of the reservoir there is a non-flow-through zone. Along the shore there exists a circulatory flow with a velocity of the order of 0.07 m/sec. The results obtained agree qualitatively and even in order of magnitude with measurements of the velocity field in Baikal. This gives grounds to suppose that one of the causes of circulation in large through-flow reservoirs may be the field of Coriolis forces.

Fig. 1

Fig. 1

Figure 1: Fig. 1

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Note: Figure translations are in progress. See original paper for figures.