ON APPROXIMATE INTEGRATION OF THE RAYLEIGH EQUATION FOR AN ARBITRARY NONLINEARITY

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V. V. KAZAKEVICH

ON APPROXIMATE INTEGRATION OF THE RAYLEIGH EQUATION FOR AN ARBITRARY NONLINEARITY

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In the present paper an approximate method is considered for integrating the Rayleigh equation

$$\ddot{x} - \mu(\dot{x} - \beta\dot{x}^3) + x = 0 \tag{1}$$

for arbitrary values of μ. The principal difficulties in solving this equation are connected with the fact that, as μ increases from zero to infinity, both the amplitudes of the periodic motions and their period grow without bound.

In the case of small μ, periodic solutions of equation (1) can be found by the Poincaré method ([1]). N. N. Bogolyubov and N. M. Krylov developed an exact method for determining the transient process and periodic solutions of nonlinear equations for small μ, which, in particular, can be used to study equation (1) ([2]). Van der Pol created an approximate method for studying transients and periodic motions for this case ([3a]). In the case of large μ, to solve equation (1) one may use the approximate method of discontinuous interpretation and the exact method of constructing asymptotic solutions ([3b], [4]).

We have developed an approximate method for determining the period of oscillations for arbitrary μ, provided that the value of the stationary amplitude is known ([5], [6]). This method, applied, for example, to the Van der Pol equation $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$, for which the value of the stationary amplitude is approximately equal to 2, gives expressions for the period $T_{\text{st}} = 2\sqrt{\pi^2 + \mu^2}$, valid for arbitrary μ. However, applying this method to equation (1) is difficult, since here the stationary amplitude depends on μ.

Let us proceed to consider an approximate method for investigating equation (1). Writing it in the general form

$$\ddot{x} + f(x, \dot{x}) = 0, \tag{2}$$

we make the change of variables (${}^5, {}^6$)

$$x = a\cos(\omega t + \gamma), \qquad \dot{x} = -a\omega\sin(\omega t + \gamma),$$

where a and γ are certain functions of time, and the angular frequency ω is a constant, which we shall determine below.

Instead of equation (2) we obtain the system

$$\begin{aligned} da/dt &= -\omega a \sin u \cos u + f(a\cos u, -a\omega\sin u)\sin u/\omega, \\ du/dt &= \omega \sin^2 u + f(a\cos u, -a\omega\sin u)\cos u/a\omega, \end{aligned} \tag{3}$$

where $u = \omega t + \gamma$.

We shall now choose ω in such a way that the mean value of du/dt in each i-th quarter of an oscillation is equal to $\omega = \omega_i$ for $a = a_0 = \text{const}$:

$$\omega_i = \frac{2}{\pi}\int_{(i-1)\pi/2}^{i\pi/2} \frac{du}{dt}\,du \qquad (i = 1, 2, 3, 4). \tag{4}$$

Carrying out the integration, we obtain

$$\omega_i^2 = \frac{4}{\pi a}\int_{(i-1)\pi/2}^{i\pi/2} f(a\cos u, -a\omega\sin u)\cos u\,du. \tag{5}$$

The duration of each of the quarters of the oscillation is $\tau_i = \pi/2\omega_i$. The period of the stationary oscillations is $T = \tau_1 + \tau_2 + \tau_3 + \tau_4$. (Expression (5) is obtained from (5′).) Let us now take the next step. Averaging $\dot{a}$ also in each of the quarters of the oscillation, we have

$$\frac{da_{i\,\text{av}}}{dt} = (-1)^i \frac{a\omega_i}{\pi} + \frac{2}{\pi\omega_i}\int_{(i-1)\pi/2}^{i\pi/2} f(a\cos u, -a\omega\sin u)\sin u\,du. \tag{6}$$

Let us now take into account that the amplitude changes Δa_i in each quarter of the oscillation are proportional not only to the magnitude $\dot{a}_i$, but also to the duration of this quarter τ_i (the latter circumstance is essential for not small μ):

$$\Delta a_i = \frac{da_{i\,\text{av}}}{dt}\tau_i = \frac{da_{i\,\text{av}}}{dt}\frac{\pi}{2\omega_i}. \tag{7}$$

The change of the amplitude during one oscillation will be determined by the expression

$$\Phi(a) \equiv \sum_{i=1}^{4} \Delta a_{i\,\mathrm{av}}.$$

The stationary amplitude is determined by the condition

$$\Phi(a) = 0. \tag{8}$$

If, as in the case of equation (1), the phase diagram is symmetric with respect to the origin, then $\omega_1 \equiv \omega_3$, $\omega_2 \equiv \omega_4$, $\dot{a}_{1\,\mathrm{av}} \equiv \dot{a}_{3\,\mathrm{av}}$, $\dot{a}_{2\,\mathrm{av}} \equiv \dot{a}_{4\,\mathrm{av}}$. For Rayleigh' s equation (1), from formulas (5) and (6) we find:

$$-\frac{\mu\beta}{\pi} a_1^2 \omega_1^3 - \omega_1^2 + \frac{2\mu\omega_1}{\pi} + 1 = 0, \tag{9a}$$

$$\frac{\mu\beta}{\pi} a_2^2 \omega_2^3 - \omega_2^2 - \frac{2\mu\omega_2}{\pi} + 1 = 0; \tag{9 }$$

$$\dot{a}_{1\mathrm{av}} = a_1 \left[\frac{1}{\pi} \left(\frac{1}{\omega_1} - \omega_1 \right) + \frac{\mu}{2} \left(1 - \frac{3\beta}{4} a_1^2 \omega_1^2 \right) \right],$$

$$\dot{a}_{2\mathrm{av}} = a_2 \left[-\frac{1}{\pi} \left(\frac{1}{\omega_2} - \omega_2 \right) + \frac{\mu}{2} \left(1 - \frac{3\beta}{4} a_2^2 \omega_2^2 \right) \right]; \tag{10}$$

$$\Phi(a) = \frac{a\pi}{2} \left[\frac{1}{\pi} \left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} \right) + \frac{\mu}{2} \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} \right) - \mu \frac{3\beta}{8} a^2 (\omega_1 + \omega_2) \right]. \tag{11}$$

We determine the periodic motion from condition (8)

$$\Phi(a) \equiv \frac{da_{1\mathrm{av}}}{dt} \frac{\pi}{2\omega_1} + \frac{da_{2\mathrm{av}}}{dt} \frac{\pi}{2\omega_2} = 0,$$

whence we obtain

$$a_{1\,\mathrm{st}} = 0 \quad \text{(equilibrium position)}, \qquad a_{2\,\mathrm{st}} = \sqrt{\frac{4}{3\beta\omega_1\omega_2} \left(1 + \frac{2}{\pi\mu} \frac{\omega_2 - \omega_1}{\omega_1\omega_2} \right)}. \tag{11a}$$

Equation (11a), together with equation (9), in which $a_1 = a_2 = a_{2\,\mathrm{st}}$ is taken, determines the frequencies ω_1 and ω_2 and the amplitude $a_{2\,\mathrm{st}}$ of the periodic motion. The stability of the stationary amplitudes is determined by the sign of $\Phi'(a_{\mathrm{st}})$: if $\Phi'(a_{\mathrm{st}}) > 0$, there is instability; if $\Phi'(a_{\mathrm{st}}) < 0$, stability. Expressions

(9) and (11a) give good accuracy for any values of μ. For $\mu = 0$, from (9) and (11) we have $\omega_{10} = \omega_{20} = 1$.

Let us consider the case of small μ, using the iteration method and taking as initial values $\omega_{10} = \omega_{20} = 1$ and $a_{\mathrm{st}0} = \sqrt{4/3\beta}$. For $\mu \neq 0$, from (9a) we have, as the next approximation,

$$\omega_{11}^2 = -\frac{\mu\beta}{\pi}\frac{4}{3\beta} + \frac{2\mu}{\pi} + 1; \qquad \omega_{11} \simeq 1 + \frac{\mu}{3\pi};$$

analogously,

$$\omega_{21} \simeq 1 - \mu/3\pi.$$

Then

$$T = \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) \simeq 2\pi\left(1 + \frac{1}{6}\frac{\mu^2}{\pi^2}\right); \qquad a_{\mathrm{st}} \simeq \sqrt{\frac{4}{3\beta}}\left(1 - \frac{2}{3\pi^2} + \frac{1}{18}\frac{\mu^2}{\pi^2}\right).$$

Let us now consider the case of finite values of μ. For convenience of calculation it is expedient to substitute (11) into (9a) and (9b). We obtain

$$-\omega_1^2\left(\frac{4\mu}{3\pi\omega_2} - \frac{8}{3\pi^2\omega_2^2} + 1\right) + \omega_1\left(\frac{2\mu}{\pi} - \frac{8}{3\pi^2\omega_2}\right) + 1 = 0, \tag{12a}$$

$$\omega_2^2\left(\frac{4\mu}{3\pi\omega_1} + \frac{8}{3\pi^2\omega_1^2} - 1\right) - \omega_1\left(\frac{2\mu}{\pi} + \frac{8}{3\pi^2\omega_1}\right) + 1 = 0. \tag{12b}$$

We see that the obtained expressions determining ω_1 and ω_2 (i.e., ultimately, the period of oscillations $T = \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)$) do not contain the parameter

Table 1

μ	τ_1	τ_2	T_{st}	$T_{\mathrm{M.U}}$	$T_{\mathrm{V.d.P}}$	a_{st}	$a_{\mathrm{M.U}}$
0	$\pi/2$	$\pi/2$	2π	2π	2π	1,86	2
1	1,5	1,9	6,8	6,7	7,2	2,02	2,17
5	2,0	4,2	12,4	11,6	–	3,34	4,3
10	3,4	7,8	22,4	19,5	20	6,0	7,5
20	6,4	15	42,8	34,7	–	11,9	14,8
∞	$0,32\mu$	$0,74\mu$	$2,12\mu$	$1,61\mu$	$1,61\mu$	$0,57\mu$	$0,66\mu$

Fig. 1

Figure 1: Fig. 1

β, on which the magnitude of the stationary amplitude depends. This means that the period of oscillations does not change when the magnitude of the stationary amplitude is changed by varying the parameter β. Thus, for any μ an important property of the systems under consideration has been established, which by existing methods can be established only for small and very large μ. The system of equations (12) can be solved with respect to ω_1 and ω_2; however, the resulting high-degree equation is difficult to solve. Therefore, from expressions (12a) and (12b), for a chosen value of μ we find ω_1 and ω_2 graphically, constructing the dependences $\omega_1 = \omega_1(\omega_2)$ and $\omega_2 = \omega_2(\omega_1)$ on the same graph and finding their points of intersection, and then determining a_{st}.

Let us consider the limiting case $\mu \to \infty$. Equations (12) give $\omega_1 \simeq 4,9/\mu$, $\omega_2 \simeq 2,1/\mu$; then from (11a) we have $a_{\text{st}} = 0,33\,\mu/\sqrt{\beta}$; for $\beta = 1/3$, $a_{\text{st}} = 0,57\,\mu$. The period $T = 2(\tau_1 + \tau_2) = 2,1\,\mu$. The exact values for $\mu \to \infty$, found by Liénard' s method, are, for $\beta = 1/3$: $T = 1,614\,\mu$, $a_{\text{st}} = 0,38\,\mu/\sqrt{\beta} = 0,659\,\mu$.

The results obtained from formulas (11a) and (12) agree sufficiently well with the exact values.

In Table 1, for various values of μ, values are given for the period T_{st} and the quarters of the oscillations τ_1 and τ_2, obtained by formula (9); for comparison with the values calculated by asymptotic formulas or by a numerical method, the results of Minorsky and Urabe (8) ($T_{\text{M.U}}$ and $a_{\text{M.U}}$) and of Van der Pol (3b) ($T_{\text{V.d.P}}$) are given. The same table gives the values of the stationary amplitude a_{st}, determined by formula (11a). We did not find in the literature the corresponding values calculated by exact formulas (except for the case of very large and very small μ). However, it follows from our work (7) that, for periodic motion, the stationary amplitude of the Rayleigh equation is numerically equal to the velocity of passage through the equilibrium position in the corresponding Van der Pol equation. The last column gives the values from (8).

A comparison of the results obtained shows a sufficiently good agreement between the approximate and exact values for arbitrary μ. In Fig. 1 the curves T_{num} and T_{pr} are the exact and approximate values of the period of oscillations, and the curves a_{num} and a_{pr} are the values of the stationary amplitude.

Let us investigate the stability of the stationary amplitudes. From condition (8) $a_{1\text{st}} = 0$ and $a_{2\text{st}}$ were found. The stationary amplitude $a_{1\text{st}}$ determines the equilibrium position, while $a_{2\text{st}}$ represents the amplitude of periodic motion.

Fig. 1

Find $\Phi'(a)$ and substitute into the expression obtained $a_{1\text{st}}$ and $a_{2\text{st}}$; if $\Phi'(a) > 0$, then the stationary amplitude is unstable; if $\Phi'(a) < 0$, then it is stable. We have

$$\Phi'(a) = \frac{\pi}{2}\left[\frac{1}{\pi}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) + \frac{\mu}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) - \frac{3\beta\mu}{8}a^2(\omega_1+\omega_2)\right] +$$

$$+\frac{a\pi}{2}(-1)\frac{3\beta}{8}\cdot 2a(\omega_1+\omega_2).$$

Substitute the value $a = a_{1\mathrm{st}} = 0$:

$$\Phi'(0) = \frac{\pi}{2}\left[\frac{1}{\pi}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) + \frac{\mu}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\right].$$

But ω_1 and ω_2 are determined by equations (9). For $a_1 = a_2 = 0$ they give:

$$\omega_1 = \frac{\mu}{\pi} + \sqrt{\frac{\mu^2}{\pi^2}+1}, \qquad \omega_2 = -\frac{\mu}{\pi} + \sqrt{\frac{\mu^2}{\pi^2}+1}.$$

Hence

$$\omega_1(0)\omega_2(0) = 1, \qquad \omega_2^2(0) - \omega_1^2(0) = -\frac{4\mu}{\pi}\sqrt{\frac{\mu^2}{\pi^2}+1},$$

$$\omega_2(0) + \omega_1(0) = 2\sqrt{\frac{\mu}{\pi^2}+1}.$$

Consequently,

$$\Phi'(0) = \frac{\pi\mu}{2}\sqrt{\frac{\mu^2}{\pi^2}+1}\left(1 - \frac{4}{\pi^2}\right) > 0.$$

Thus the equilibrium position is unstable for all μ. Further,

$$\Phi'(a_{2\mathrm{st}}) = -\frac{3\beta\mu(\omega_1+\omega_2)}{4}\,a_{\mathrm{st}2}^2\,\frac{\pi}{2} < 0.$$

Consequently, the periodic motion is stable for all μ.

In conclusion, we note that the method presented here is applicable to the investigation of the equation $x'' + \mu\varphi(\dot{x}) + x = 0$, where $\varphi(\dot{x})$ is a polynomial in $\dot{x}$.

Moscow Polygraphic Institute

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