THE PROBLEM E FOR ONE CLASS OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Abstract

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Let D be a finite simply connected domain bounded by the segment AB of the x-axis and by a Jordan curve σ, lying in the half-plane $y > 0$ and with endpoints at A and B. The equation

$$Gu \equiv y^m u_{yy} + u_{xx} - f(x, y, u, u_x, u_y) = 0, \tag{1}$$

where $m > 0$, and f is a given function of its arguments, is elliptic for $y > 0$, while on the line $y = 0$ it degenerates parabolically.

In [1] the solvability of the Dirichlet problem for equation (1) was investigated. In the present paper the problem E (see [2]) for equation (1) is studied.

Problem E. Find a twice continuously differentiable in the domain D solution of equation (1), bounded in the domain D, continuous up to σ, and taking prescribed continuous values on σ.

Without loss of generality, we shall assume that zero values are prescribed on σ.

Suppose that the function f and the curve σ satisfy the following conditions:

Condition A. 1) f is continuously differentiable with respect to its arguments, and $-f_u \le -k < 0$ for $(x, y) \in D$ and all values of u, u_x, u_y; 2) for $(x, y) \in D$, $|u| \le N$, $-\infty < u_x, u_y < \infty$, the estimates $|f_u| \le M$, $|f_{u_x}| \le M$, $|f_{u_y}| \le M$ hold, where $M(N)$ is some positive number.

Condition B′. a) For $m = 1$, $-\partial f/\partial u_y \ge 1$ in the domain

$$\mathfrak{R}_h = \{(x, y) \in D - \overline{D}_h,\ |u| \le N,\ -\infty < u_x, u_y < \infty\},$$

where

$$D_h = \{(x, y) \in D,\ y > h\},$$

and h is a sufficiently small positive number; b) for $1 < m < 2$, $-\partial f/\partial u_y > y^{m-1}$ in $\mathfrak{R}_h$; c) for $m \ge 2$, $-\partial f/\partial u_y > 0$ in $\mathfrak{R}_h$.

Condition C. Let the curve σ at the points $A(a, 0)$ and $B(b, 0)$ $(a < b)$ end in arbitrarily small arcs AA' and BB', whose equations may be given in the form $y = l(x)$, where $l'(x) \geq 0$ on AA' and $l'(x) \leq 0$ on BB'. The remaining part σ' of the curve σ is such that, for the σ_0'-image of σ' under the mapping $\xi = x$, $t = 2\sqrt{y}$, at each point $\zeta \in \sigma_0'$ there exists a certain circle S whose intersection with the closed domain $\overline{D}_0$ consists of just one point ζ, $S \cap \overline{D}_0 = \zeta$, where D_0 is the image of the domain D.

Theorem. If the function f satisfies conditions A and B$'$, and the curve σ satisfies condition C, then the solution of problem E for equation (1) exists and is unique.

We note that, under conditions A and B$'$, for the solution $u(x, y)$ of problem E the estimate

$$|u| \leq \frac{1}{k} \max_{\overline{D}} |f_0(x, y)| \equiv \mu, \tag{2}$$

holds, where k is the number from condition A, and $f_0(x, y) = f(x, y, 0, 0, 0)$.

The uniqueness of the solution of problem E follows from the following lemma:

Lemma. Let the linear equation

$$y^m u_{yy} + u_{xx} + a u_x + b u_y + c u = F(x, y) \tag{3}$$

with coefficients bounded in D, where $c \geq 0$ and the conditions are fulfilled: a) for $m = 1$, $b \geq 0$ in $D - D_h$; b) for $1 < m < 2$, $b > y^{m-1}$ in $D - D_h$; c) for $m \geq 2$, $b > 0$ in $D - D_h$.

Then the solution of problem E for equation (3) is unique.

This lemma is proved analogously to the uniqueness lemma in [2]. The proof of the existence of a solution is carried out according to the scheme of [1], which is a certain modification of Perron’ s method [3].

Let a continuous function $\varphi(x, y)$ be given in the closed domain $D + \partial D$, and let a disk $K \subset D$ be given. By $\mathfrak{M}_K(\varphi)$ we denote the function which, outside the disk K and on its boundary ∂K, coincides with φ, while inside K it is the solution of equation (1) constructed from the boundary values φ on ∂K.

The function φ is called a supersolution (subsolution) if for every disk $K \subset D$ one has $\varphi \geq \mathfrak{M}_K(\varphi)$ $[\varphi \leq \mathfrak{M}_K(\varphi)]$. A supersolution (subsolution) φ is called an upper (lower) function if $\varphi \geq 0$ $(\varphi \leq 0)$ on ∂D.

From condition C it follows that for each point $(x_0, y_0) \in \sigma$ (we consider σ closed) there exists some point $(\bar{x}, \bar{y})$ $(\bar{y} \geq 0)$ such that for any point $(x, y) \in D$

$$r^2 = (x - \bar{x})^2 + 4(\sqrt{y} - \sqrt{\bar{y}})^2 > R^2 = (x_0 - \bar{x})^2 + 4(\sqrt{y_0} - \sqrt{\bar{y}})^2.$$

The function

$$w(x, y; x_0, y_0) = \mu \left[1 - (1 + R^2)^p (1 + r^2)^{-p}\right],$$

where μ is the number from formula (2), for a suitable choice of the parameter $p > 0$, satisfies the conditions: 1) $w(x, y; x_0, y_0) > 0$ in D, 2) $w(x_0, y_0; x_0, y_0) = 0,\ (x_0, y_0) \in \sigma$, 3) $Gw \leq 0$ in D.

Using $w(x, y; x_0, y_0)$, we form new functions

$$\varphi_0(x, y) = \varepsilon/2 + w(x, y; x_0, y_0), \qquad \psi_0(x, y) = -\varepsilon/2 - w(x, y; x_0, y_0),$$

which will be, respectively, upper and lower functions for equation (1), and $\varphi_0(x, y) = \varepsilon/2,\ \psi_0(x, y) = -\varepsilon/2$.

Since φ_0 and ψ_0 are continuous functions, there is a certain neighborhood $\mathfrak{A}_0 \subset \sigma$ of the point $P_0(x_0, y_0)$ such that $\varphi_0 - \psi_0 < 2\varepsilon$ for all $(x, y) \in \mathfrak{A}_0$. Choosing, by the Heine-Borel lemma, a finite number of neighborhoods $\mathfrak{A}_i \subset \sigma$ $(i = 0, 1, \dots, n)$ covering the whole curve σ, we obtain upper functions $\varphi_0, \varphi_1, \dots, \varphi_n$ and lower functions $\psi_0, \psi_1, \dots, \psi_n$ such that $\varphi_i - \psi_i < 2\varepsilon$ for $(x, y) \in \mathfrak{A}_i$. Then the functions $\varphi = \min(\varphi_0, \varphi_1, \dots, \varphi_n)$ and $\psi = \max(\psi_0, \psi_1, \dots, \psi_n)$ are, respectively, upper and lower functions, and $\varphi - \psi < 2\varepsilon$ for $(x, y) \in \sigma$. Hence, in view of the arbitrariness of $\varepsilon > 0$, it follows that the lower envelope of the upper functions and the upper envelope of the lower functions coincide everywhere on σ.

We shall show that the lower envelope of the upper functions and the upper envelope of the lower functions coincide also inside the domain. For this it is enough to construct some upper function $\widetilde{\varphi}$ and lower function $\widetilde{\psi}$ for which at each point $(x, y) \in D$, $\widetilde{\varphi} - \widetilde{\psi} < \varepsilon$ for any $\varepsilon > 0$. Denote by Λ the maximum of the difference $\varphi - \psi$ in the domain D, where φ and ψ are the functions constructed above.

We construct an upper function $\varphi^{(1)}$ and a lower function $\psi^{(1)}$ such that, for an arbitrary domain $D_1,\ \overline{D}_1 \subset D$,

$$\max_{\overline{D}1} (\varphi^{(1)} - \psi^{(1)}) < q\Lambda, \tag{4}$$

where $0 < q < 1$ is a certain number independent of Λ.

For some disk $K_i \subset D$ we form the functions $\varphi_i = \mathfrak{M}_{K_i}(\varphi)$ and $\psi_i = \mathfrak{M}_{K_i}(\psi)$. We shall show that

$$\max_{L_i} (\varphi_i - \psi_i) < q_i \Lambda, \tag{5}$$

where L_i is a disk concentric with K_i, with $L_i \subset K_i$, and $0 < q_i < 1$ is a number depending only on L_i.

In the disk K_i the function $\chi = \varphi_i - \psi_i$ satisfies the equation

$$F(\chi) \equiv y^m \frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial x^2} - \frac{\partial \bar{f}}{\partial \chi_x} \frac{\partial \chi}{\partial x} - \frac{\partial \bar{f}}{\partial \chi_y} \frac{\partial \chi}{\partial y} - \frac{\partial \bar{f}}{\partial \chi} \chi = 0,$$

where the notation adopted is

$$\bar{g} = g\left\{x, y, \psi_i + \theta(\varphi_i - \psi_i), \frac{\partial}{\partial x}[\psi_i + \theta_1(\varphi_i - \psi_i)], \frac{\partial}{\partial y}[\psi_i + \theta_2(\varphi_i - \psi_i)]\right\}, \quad 0 < \theta, \theta_1, \theta_2 < 1.$$

Consider the function

$$\tilde{\chi} = \{\exp a(r^2 - R^2) + \lambda[1 - (1+y)^{-\nu}]\}\Lambda,$$

where $r^2 = (x - x_1)^2 + (y - y_1)^2$, (x_1, y_1) is the center, R is the radius of the disk K_i, and $0 < \lambda = \text{const} < 1$. Choosing sufficiently small $a > 0$ and sufficiently large $\nu > 0$, one can ensure that $F\tilde{\chi} \leqslant 0$. Then choose λ from the inequality

$$\exp a(\rho^2 - R^2) + \lambda < 1, \tag{6}$$

where ρ is the radius of the disk L_i. Since inside the disk K_i, $F\tilde{\chi} \leqslant 0$, and on ∂K_i $\varphi_i - \psi_i < \tilde{\chi}$, it follows that everywhere in the disk K_i $\varphi_i - \psi_i \leqslant \tilde{\chi}$. Consequently,

$$\max_{L_i}(\varphi_i - \psi_i) < \max_{L_i} \tilde{\chi} \leqslant \max_{\partial L_i} \tilde{\chi}$$

$$= \{\exp a(\rho^2 - R^2) + \lambda[1 - (1+Y)^{-\nu}]\}\Lambda = q_i\Lambda,$$

where Y is the maximum of the ordinates of points of ∂L_i. Hence, and from (6), (5) follows.

Take a finite covering of the domain $\bar{D}_1$ by disks $L_i \subset D$ $(i = 1, 2, \dots, l)$ and construct the functions φ_i and ψ_i. For each function φ_i in ψ_i inside the disk L_i we obtain inequality (5).

The functions $\varphi^{(1)} = \min(\varphi_1, \dots, \varphi_l)$ and $\psi^{(1)} = \max(\psi_1, \dots, \psi_l)$ will be, respectively, an upper and a lower function. Since every point of the domain $\bar{D}_1$ belongs to some disk L_j, we have

$$\max_{\bar{D}_1}(\varphi^{(1)} - \psi^{(1)}) \leqslant \max_{L_j}(\varphi_j - \psi_j) < q_j\Lambda \leqslant q\Lambda,$$

where $q = \max(q_1, \dots, q_l)$.

Thus we have constructed functions $\varphi^{(1)}$ and $\psi^{(1)}$ satisfying inequality (4). Continuing this process, we construct a sequence of upper functions $\varphi^{(1)}, \varphi^{(2)}, \ldots$ and lower functions $\psi^{(1)}, \psi^{(2)}, \ldots$, whose difference $\varphi^{(s)} - \psi^{(s)}$ is everywhere in the domain D_1 less than $q^s \Lambda$.

Since $q < 1$ and D_1 is an arbitrary subdomain of the domain D, we obtain that the lower envelope of the upper functions and the upper envelope of the lower functions coincide everywhere in the domain D. Denote this common envelope by $v(x, y)$.

We shall prove that $v(x, y)$ is a solution of equation (1). To this end it suffices to show that for any disk $K \subset D$,

$$v = \mathfrak{M}_K(v).$$

Using the constructed upper functions $\varphi^{(1)}, \varphi^{(2)}, \ldots$, form the functions

$$v_n = \mathfrak{M}_K(\varphi^{(n)}).$$

Outside the disk K and on ∂K, the sequence $\{v_n\}$ converges uniformly to the function v, while inside K it is a sequence of solutions of equation (1) converging uniformly on the boundary ∂K. By Harnack' s theorem (see, for example, (1)), the sequence $\{v_n\}$ converges uniformly inside K to the solution of equation (1) taking the value v on ∂K. From the continuity of this limiting function in $K + \partial K$ it follows that

$$v = \mathfrak{M}_K(v).$$

The boundedness of the solution in the domain D follows from the inequality

$$\psi(x, y) \leqslant v(x, y) \leqslant \varphi(x, y),$$

where φ and ψ are some upper and lower functions.

Remark. The uniqueness of the solution of problem E holds without the additional requirement of condition C on the curve σ.

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Note: Figure translations are in progress. See original paper for figures.

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