EXISTENCE OF A CONVEX SURFACE WITH A PRESCRIBED FUNCTION OF THE PRINCIPAL RADII OF CURVATURE

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Abstract

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EXISTENCE OF A CONVEX SURFACE WITH A PRESCRIBED FUNCTION OF THE PRINCIPAL RADII OF CURVATURE

The subject of the present article is the solution of the question of the existence of a closed convex surface satisfying the condition

$$f(R_1, R_2) = \varphi(n), \tag{1}$$

where R_1 and R_2 are the principal radii of curvature, and n is the unit normal vector to the surface. This problem was considered by Christoffel in the case when $f = R_1 + R_2$, and by Minkowski when $f = R_1R_2$.

1. The formulation of the final result is somewhat complicated. Therefore we shall postpone it to the end of the article and begin with the proof of the existence of a surface satisfying condition (1), subjecting the functions f and φ to the corresponding restrictions in the course of the proof. First of all, we shall assume that the function f is defined for all positive values of R_1, R_2, is symmetric ($f(R_1, R_2) = f(R_2, R_1)$), and is strictly increasing, i.e.

$$\partial f/\partial R_1 > 0, \qquad \partial f/\partial R_2 > 0. \tag{*}$$

As for the function φ, we assume it to be even ($\varphi(n) = \varphi(-n)$). Both functions f and φ are initially assumed to be analytic. At the end this requirement will be weakened to twice differentiability.

2. A convex surface F is determined by condition (1) uniquely up to parallel translation. If $\varphi(n)$ is an even function, as we assume, then the surface F has a center of symmetry ([1]).

Let the surface F undergo an infinitesimal centrally symmetric deformation into a surface F_t with support function $H_t = H + tZ$, where H is the support function of F. Then, if the function f for the surface F_t is stationary at $t = 0$, i.e. $df(R_1(t), R_2(t))/dt = 0$, then $Z \equiv 0$. We shall need this result for the case when the surface F and its deformation are analytic, and in this form it is essentially contained in the work of A. D. Aleksandrov ([2]).

3. Let $H(n)$ be the support function of the surface F, satisfying equation (1), on the unit sphere ω. If the center of the surface is taken as the origin of coordinates, then $H(n)$ will be an even function ($H(n) = H(-n)$). It satisfies an equation of elliptic type $\Phi(H_{11}, H_{12}, \dots, H, n) = 0$, which is obtained from condition (1). Ellipticity follows from the monotonicity of f with respect to the variables R_1 and R_2. The corresponding equation in variations has the form

$$\frac{\partial \Phi}{\partial H_{11}} p_{11} + \frac{\partial \Phi}{\partial H_{12}} p_{12} + \dots + \frac{\partial \Phi}{\partial H} = L(p) = 0. \tag{2}$$

As indicated in item 2, the equation $L(p) = 0$ in the class of even functions p ($p(n) = p(-n)$) has no solutions other than the trivial one $p \equiv 0$. The linear operator L induces on the unit sphere a Riemannian metric

$$ds^2 = \frac{\partial \Phi}{\partial H_{11}} du_1^2 + \frac{\partial \Phi}{\partial H_{12}} du_1 du_2 + \frac{\partial \Phi}{\partial H_{22}} du_2^2,$$

where u_1, u_2 are curvilinear coordinates on the sphere. In view of the symmetry of the surface F, and consequently of the evenness of H, the mapping of the sphere ω onto itself by symmetry with respect to its center is isometric in the metric ds^2.

The problem of reducing equation (2) on the sphere ω to the canonical form

$$\Delta p + A p_{u_1} + B p_{u_2} + C u = 0, \tag{3}$$

where Δ is the second Beltrami differential parameter, is, as is known, connected with the conformal mapping of the manifold with metric ds^2 onto the sphere. In view of the above-mentioned symmetry of the metric of the manifold M, one may assume that the conformal mapping preserves this symmetry.

4. We shall solve the problem of constructing a surface satisfying condition (1) by the method of continuous continuation with respect to a parameter. To this end we include the function $\varphi(n)$ in a continuous family $\varphi_\lambda(n)$, putting

$$\varphi_\lambda(n) = \lambda \varphi + (1 - \lambda) f(1, 1), \quad 0 \leq \lambda \leq 1.$$

The problem is trivially solvable for $\lambda = 0$. The corresponding surface is the sphere of unit radius. Suppose that the problem is solvable for the function φ_{λ_0}. We shall show that it is then solvable for any function φ_λ when λ is sufficiently close to λ_0.

The equation for the support function $p_\lambda(n)$ of the surface F_λ may be represented in the form

$$L(\bar{p}) + R(\bar{p}) + (\lambda - \lambda_0)(\varphi(n) - f(1,1)) = 0, \tag{4}$$

where $\bar{p} = p_\lambda - p_{\lambda_0}$, L is the linear elliptic operator (3) corresponding to the surface F_{λ_0}, and R is a quadratic expression with respect to the function $\bar{p}$ and its derivatives up to the second order.

Passing from equation (4) to an integral equation with the aid of an even fundamental solution of the equation $\Delta\bar{p} = 0$ with a logarithmic singularity at two diametrically opposite points of the sphere, we then obtain

$$\bar{p} + \Omega\bar{p} = A, \tag{5}$$

where

$$\Omega(\bar{p}) = \int_\omega K(n, n')\bar{p}\, d\omega, \qquad A = \int_\omega K_1(n, n') R\, d\omega.$$

The kernels K and K_1 are even in both variables. As $|n - n'| \to 0$,

$$K \sim 1/|n - n'|, \qquad K_1 \sim \ln|n - n'|.$$

5. To solve equation (5) we shall apply the method of successive approximations. In this connection denote by $C'_{2,\alpha}$ the space of twice differentiable even (in n) functions on the unit sphere whose second derivatives satisfy a Hölder condition with exponent $\alpha > 0$. The operator Ω, acting in this space, is completely continuous. Since the homogeneous equation $\bar{p} + \Omega\bar{p} = 0$ has no solution in $C'_{2,\alpha}$ except the trivial one (Sec. 3), equation (5) is uniquely solvable in $C'_{2,\alpha}$ for any right-hand side A also belonging to $C'_{2,\alpha}$.

We now define the successive approximations $\bar{p}_k$ by means of the recurrent system

$$\bar{p}_k + \Omega\bar{p}_k = A(p_{k-1}).$$

As the initial approximation take $\bar{p}_0(n) \equiv 0$. The process of successive approximations converges when λ is sufficiently close to λ_0 and gives the solution of our problem for such λ. This solution, by virtue of the analyticity and ellipticity of the original equation, will be analytic.

6. We shall now show that, under certain conditions on the functions f and φ, the problem under consideration is solvable for any λ. For this, obviously, it is enough to guarantee the existence of a priori estimates for the function $p_\lambda(n)$ and its derivatives up to the second order. In view of the closedness and con-

convexity of the surface F_λ, this is equivalent to the existence of positive a priori estimates for the principal radii of curvature. In the author' s paper ([3]) the existence of such estimates was proved under the following conditions. Suppose that, when differentiating the function φ along an arc of a great circle on the unit sphere,

$$a \leq \varphi'^2 \leq b, \qquad A \leq \varphi'' \leq B.$$

Then, for the existence of positive a priori estimates for the principal radii of curvature of the surface F_λ, it is sufficient that, for any constants α, β, λ such that $a \leq \alpha \leq b,\ A \leq \beta \leq B$, and $0 \leq \lambda \leq 1$, the inequalities

$$\lim_{\substack{R_2 = R_2(R_1, n) \\ R_1 \to \infty}} \left\{ (R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\lambda^2 \alpha}{(\partial f / \partial R_2)^2} \right\} < \beta\lambda,$$

$$\lim_{\substack{R_1 = R_1(R_2, n) \\ R_2 \to 0}} \left\{ (R_1 - R_2) \frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2} \frac{\lambda^2 \alpha}{(\partial f / \partial R_1)^2} \right\} > \beta\lambda. \qquad (**)$$

After the problem has been solved for the case of analytic functions f and φ, the analyticity condition can be weakened to the requirement of twice differentiability. For this it is enough to approximate the functions f and φ by analytic ones and, after solving the problem, pass to the limit in the solution. As a result, the following theorem is obtained.

Theorem. *Let $f(R_1, R_2)$ be a twice differentiable, symmetric function strictly increasing in both variables. Then, for any even function φ, provided conditions $(**)$ are fulfilled, there exists a closed convex surface satisfying the condition*

$$f(R_1, R_2) = \varphi(n),$$

where R_1, R_2 are the principal radii of curvature of the surface, and n is the unit vector of the outer normal.

In conclusion, we note that an analogous result can also be obtained in a more general case, when the function f also depends on n, being an even function of this variable. The corresponding equation has the form

$$f(R_1, R_2, n) = \varphi(n).$$

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CITED LITERATURE

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2. A. D. Aleksandrov, *DAN*, **22**, No. 3 (1939).
3. A. V. Pogorelov, *DAN*, **174**, No. 6 (1967).

Note: Figure translations are in progress. See original paper for figures.