A GENERALIZED M. RIESZ FORMULA AND EXTENSION OF CONVOLUTION

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Abstract

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MATHEMATICS

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A GENERALIZED M. RIESZ FORMULA AND EXTENSION OF CONVOLUTION

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1°. Let Π_n denote the set of all trigonometric polynomials of order $\leqslant n$. Let

$$\Phi(t) = \sum_{k=0}^{n} r_k \sin(kt + \alpha_k). \tag{1}$$

The polynomial

$$\widetilde{\Phi}(t) = r_n + 2\sum_{k=0}^{n-1} r_k \cos[(n-k)t + \alpha_n - \alpha_k] \tag{2}$$

will be called the one associated with the polynomial $\Phi(t)$. By $\widetilde{C}$ and $\widetilde{L}_r$ we shall denote, respectively, the space of all 2π-periodic continuous functions and the space of all 2π-periodic functions summable with the r-th power. Any of the spaces mentioned will be denoted by the letter E^*. Clearly, Π_n may be regarded as a subspace of the space E. Put

$$\sigma(f,x) = \int_0^{2\pi} f(x+\theta)\Phi(\theta)\,d\theta; \tag{3}$$

$$\overline{U}(f,x) = \frac{\pi}{2n}\sum_{r=1}^{2n} f\left(x+\varphi_r - \frac{\alpha_n}{n}\right)(-1)^{r-1}\widetilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right), \qquad \varphi_r = \frac{2r-1}{2n}\pi. \tag{4}$$

In (2) it was established that, for $t \in \Pi_n$,

$$\sigma(t,x) = \overline{U}(t,x), \qquad -\infty < x < \infty. \tag{5}$$

Identity (5) will be called the generalized M. Riesz identity, since the M. Riesz formula ([3], [4]) follows from it; with the aid of this formula one obtains a very simple proof of the classical theorem of S. N. Bernstein, according to which

$$\|t'\| \leqslant n\|t\|.$$

The present note is devoted mainly to the application and generalization of identity (5).

2°. We shall consider the convolution (3) as a linear operation from Π_n into Π_n, where Π_n is considered as a subspace of the space E. Denote the norm of this operation by τ_n. Thus:

$$\tau_n = \sup_{\|t\|<1,\ t\in\Pi_n} \|\sigma(t)\|.$$

We pose the following problem. Under what conditions does there exist a linear operation U from E into E, satisfying the conditions:

$$\begin{aligned} &1)\quad U(t) = \sigma(t), \quad t \in \Pi_n; \\ &2)\quad \|U\|_E^E = \tau_n, \quad \text{where } \|U\|_E^E \text{ is the norm of the operation } U \text{ from } E \text{ into } E. \end{aligned} \tag{6}$$

* Some theorems of this note are valid for the spaces E from ([1]).

Such an operation U we call an **extension of the convolution σ from Π_n to** E. It is well known that in the case of linear functionals this problem always has a positive solution.

Denote by Ω_n^Φ the set of all linear operations U from E into E satisfying condition (6). Introduce the quantity

$$\rho_n = \rho_n^\Phi(E) = \inf_{U\in\Omega_n^\Phi} \|U\|.$$

Theorem 1. Let the linear operation U_0 be an extension of the convolution (3) from Π_n to E. Then

$$\|U_0\| = \rho_n = \tau_n. \tag{7}$$

Proof. Since $U_0 \in \Omega_n^\Phi$, we have $\|U_0\| \geq \rho_n$. By assumption, $\|U_0\|_E^E = \tau_n$; hence

$$\tau_n \geq \rho_n. \tag{8}$$

On the other hand, for any $U \in \Omega_n^\Phi$,

$$\|U\| \geq \tau_n, \tag{9}$$

because U satisfies equality (6). In view of (9), we have

$$\tau_n \leq \rho_n. \tag{10}$$

From inequalities (8), (10) we conclude that (7) holds.

Theorem 2. If the kernel (1) is nonnegative, $\Phi(t) \geq 0$, $-\infty < t < \infty$, then the convolution (3) itself is its own extension from Π_n to E. Moreover,

$$\tau_n = \int_0^{2\pi} \Phi(t)\, dt.$$

Proof. It is obvious that

$$\tau_n \geq \|\sigma(1)\| = \int_0^{2\pi} \Phi(t)\, dt. \tag{11}$$

On the other hand, by virtue of the nonnegativity $\Phi(t) \geq 0$ *, $-\infty < t < \infty$,

$$\|\sigma(f)\| = \left\| \int_0^{2\pi} f(x+t)\Phi(t)\, dt \right\| \leq \int_0^{2\pi} \Phi(t)\, dt\, \|f\|.$$

Therefore

$$\tau_n \leq \|\sigma\|_E^E \leq \int_0^{2\pi} \Phi(t)\, dt. \tag{12}$$

From (11) and (12) it follows

Theorem 3. In the space $\widetilde{L}_2$, for a kernel $\Phi(t)$ of arbitrary sign, the convolution (3) itself is its own extension from Π_n to L_2. Moreover,

$$\tau_n = \pi r_{j_0}, \qquad r_{j_0} = \max\left\{ \max_{j=1,2,\dots,n} r_j,\ 2r_0|\sin\alpha_0| \right\}.$$

Proof. It is easy to see that

$$\left\| \sigma\left(\frac{\cos kx}{\|\cos kx\|_{\widetilde{L}_2}} \right) \right\| = \begin{cases} |2\pi r_0 \sin\alpha_0|, & k = 0, \\ \pi r_k, & k = 1, 2, \dots, n. \end{cases}$$

Therefore

$$\|\sigma\|_{\widetilde{L}_2}^{\widetilde{L}_2} \geq \tau_n \geq \pi r_{j_0}. \tag{13}$$

On the other hand, from Parseval' s equality it follows that

$$\tau_n \leq \|\sigma\|_{\widetilde{L}_2}^{\widetilde{L}_2} \leq \pi r_{j_0}. \tag{14}$$

(13) and (14) imply Theorem 3.

* We assume that $\|1\| = 1$. The legitimacy of passing to the norm under the integral sign is easy to justify.

By virtue of Theorems 2 and 3 one might think that in any space E the convolution (3) is always its extension from Π_n to E. In fact this is not so. In the case of the spaces $\widetilde{C}$ and $\widetilde{L}_1$ the situation changes substantially. This is seen from the following theorem.

Theorem 4. Let $E = \widetilde{C}$ or $E = \widetilde{L}_1$, and let the kernel $\Phi(t)$ be such that

$$\widetilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right) \geqslant 0, \qquad r = 1, 2, \ldots, 2n. \tag{15}$$

Then the extension of the convolution (3) from Π_n to $\widetilde{C}$, or from Π_n to $\widetilde{L}_1$*, is the operator $\overline{U}(f, x)$, which is given by formula (4). Moreover, $\tau_n = \pi r_n$.

Proof. For definiteness we shall consider the case $E = \widetilde{C}$. By identity (5) it is necessary only to show that $\|U\|_{\widetilde{C}}^{\widetilde{C}} = \tau_n$. Let $t_n^*(x) = \sin(nx + \alpha_n)$. One can verify that

$$\overline{U}(t_n^*, 0) = \frac{\pi}{2n} \sum_{r=1}^{2n} \widetilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right) = \pi r_n.$$

Consequently,

$$\|\overline{U}\|_{\widetilde{C}}^{\widetilde{C}} \geqslant \tau_n \geqslant \pi r_n. \tag{16}$$

On the other hand, from (4) it follows that for any $f \in \widetilde{C}$

$$\|U(f)\| \leqslant \frac{\pi}{2n} \sum_{r=1}^{2n} \left|\widetilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right)\right|.$$

Hence, by inequalities (15), we obtain that

$$\|U\|_{\widetilde{C}}^{\widetilde{C}} \leqslant \pi r_n. \tag{17}$$

From (16) and (17) we conclude that $\|\overline{U}\|_{\widetilde{C}}^{\widetilde{C}} = \tau_n = \pi r_n$.

Theorem 5. If the kernel $\Phi(t)$ is the derivative of order k of the Dirichlet kernel

$$\Phi(t) = D_n^{(k)}(t), \qquad k = 1, 2, \dots, \tag{18}$$

then the associated kernel $\widetilde{\Phi}(t)$ is nonnegative on the whole real axis.

We indicate the proof. Since

$$D_n^{(k)}(t) = \sum_{\nu=1}^{n} \nu^k \sin\left(\nu t + \frac{k+1}{2}\pi\right), \qquad k = 1, 2, \dots,$$

then, according to (2),

$$\widetilde{D}_n^{(k)}(t) = n^k + 2\sum_{\nu=1}^{n-1} \nu^k \cos(n-\nu)t, \qquad k = 1, 2, \dots.$$

Applying L. Fejér' s theorem (5), we see that

$$\widetilde{D}_n^{(k)}(t) \geqslant 0, \qquad -\infty < t < \infty, \qquad k = 1, 2, \dots.$$

From Theorems 4 and 5 there follows

Corollary 1. If $\Phi(t)$ is defined according to equality (18), then the extension of the convolution (3) from Π_n to $\widetilde{C}$, or from Π_n to $\widetilde{L}_1$, is the operator (4), where $\tau_n = \pi n^k$.

3°. Equality (5) has so far been considered only for the case when $f \in \Pi_n$. If it is considered for arbitrary $f \in \widetilde{C}$, then the following holds.

Theorem 6. For any $f \in \widetilde{C}$ there exists a set M, consisting of at least $(2n+1)$ distinct points of $[0, 2\pi)$, such that

$$\overline{U}(f, x) = \sigma(f, x), \qquad x \in M.$$

If $f \in \Pi_n$, then $M = (-\infty, \infty)$.

* We assume that $\widetilde{L}_1$ consists of everywhere finite functions.

We outline the proof. By virtue of identity (5) and the linearity of the operators (3) and (4), we have

$$\overline{U}(f) - \sigma(f) = \overline{U}(f - s_n(f)) + \sigma(f - s_n(f)), \tag{19}$$

where $s_n(f)$ is the partial sum of order n of the Fourier series of f. Since Φ is a polynomial of order n, $\sigma(f - s_n(f)) = 0$. Therefore, by (19),

$$\overline{U}(f) - \sigma(f) = \overline{U}(f - s_n(f)). \tag{20}$$

Let us note that

$$\int_0^{2\pi} \overline{U}(f - s_n(f), x) e^{ikx}\, dx = 0, \qquad k = 1, 2, \dots, n.$$

Consequently[6], $\overline{U}(f - s_n(f), x)$ has in the interval $[0, 2\pi)$ no fewer than $(2n+1)$ distinct zeros. We now apply equality (20). It is obvious that identity (5) is a consequence of Theorem 6.

Theorem 7. *If inequalities (15) are satisfied, then for every $f \in \widetilde{C}$ there exists a set M, consisting of at least $(2n+1)$ distinct points of $[0, 2\pi)$, such that*

$$|\sigma(f, x)| \le \pi r_n \sup_{\varphi_r} \left| f\left(x + \varphi_r - \frac{\alpha_n}{n}\right) \right|, \qquad x \in M.$$

If $f \in \Pi_n$, then $M = (-\infty, \infty)$.

We outline the proof. According to Theorem 6,

$$|\sigma(f, x)| \le \sup_{\varphi_r} \left| f\left(x + \varphi_r - \frac{\alpha_n}{n}\right) \right| \frac{\pi}{2n} \sum_{r=1}^{2n} \left| \widetilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right) \right|, \qquad x \in M. \tag{21}$$

Since inequalities (15) are satisfied, the sum on the right-hand side of (21) is equal to $2nr_n$.

Corollary 2. *For any $f \in \widetilde{C}$ there exists a set M, consisting of at least $(2n+1)$ distinct points of $[0, 2\pi)$, such that*

$$|s_n^{(k)}(f, x)| \le n^k \sup_{r=1,2,\dots,2n} \left| f\left(x + \frac{2r - n - 2}{2n}\pi\right) \right|, \qquad x \in M, \quad k = 1, 2, \dots.$$

If $f \in \Pi_n$, then $M = (-\infty, \infty)$.

This assertion follows from Theorems 7 and 5, when $\Phi(t) = D_n^{(k)}(t)$.

Theorem 8. *Let $f(x)$ be continuous on $[-1,1]$. Then there exists a set M, consisting of at least n points of $(-1,1)$, such that*

$$\left|\frac{d}{dx} s_n[f(\cos\theta), \arccos x]\right| \leq \frac{n}{\sqrt{1-x^2}} \sup_{\varphi_r} |f[\cos(\arccos x + \varphi_r)]|, \qquad x \in M. \tag{22}$$

If f is a polynomial of degree n, then $M = (-1,1)$.

It is clear that the most interesting case of (22) is the known inequality of S. N. Bernstein[7], according to which

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}} \max_{-1\leq x\leq 1} |P(x)|, \qquad -1 < x < 1,$$

if $P(x)$ is an algebraic polynomial of degree n.

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Note: Figure translations are in progress. See original paper for figures.