GENERAL BOUNDARY CONDITIONS FOR MARKOV PROCESSES WITH A COUNTABLE SET OF STATES

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Abstract

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MATHEMATICS

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GENERAL BOUNDARY CONDITIONS FOR MARKOV PROCESSES WITH A COUNTABLE SET OF STATES

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1. Markov processes in a countable phase space E are conveniently specified by transition-probability densities $a(x, y)$ $(x, y \in E)$. We shall assume that $|a(x, x)| < \infty$ and $\sum_y a(x, y) \equiv 0$. The densities $a(x, y)$ determine the course of the process up to the time τ of the first accumulation of jumps. Our problem is to describe all ways of continuing the process after time τ.

In the language of semigroup theory the problem may be formulated as follows. Let C be the space of bounded functions on E with norm $\|f\| = \sup|f(x)|$. Denote by $\mathfrak{A}$ the operator defined by the formula $\mathfrak{A}f(x) = \sum_y a(x, y)f(y)$ on the domain $\mathcal{D}$, consisting of all functions f such that $f \in C$ and $\mathfrak{A}f \in C$. We shall call a one-parameter semigroup T_t of linear operators in C an $\mathfrak{A}$-semigroup if it satisfies the conditions: a) $T_t f \geq 0$ for $f \geq 0$; b) $\|T_t f\| \leq \|f\|$; c) the weak infinitesimal operator A of the semigroup T_t is a restriction of the operator $\mathfrak{A}$; d) $\lim_{t \downarrow 0} T_t f(x) = f(x)$ for all $x \in E$. It is required to describe all $\mathfrak{A}$-semigroups.

This problem is solved under the assumption that:

Condition F. For some $\lambda > 0$, the equation $\mathfrak{A}f = \lambda f$ has only a finite number of linearly independent nonnegative bounded solutions.

Restrictions A of the operator $\mathfrak{A}$ are determined by conditions which are conveniently formulated in terms of Martin boundaries: the exit boundary B and the entrance boundary $\hat{B}$. Condition F is equivalent to the assumption that the active minimal part B_0 of the exit boundary B is finite.

The general form of the boundary conditions describing the restrictions of the operator $\mathfrak{A}$ of interest to us is given by formula (1) (see below).

For the same class of operators $\mathfrak{A}$, W. Feller studied (1) $\mathfrak{A}$-semigroups satisfying the additional requirement: the infinitesimal operator of the adjoint semigroup is a restriction of the operator $\mathfrak{A}^*\eta(y) = \sum_x \eta(x)a(x, y)$. The boundary conditions found by Feller constitute a special case of our conditions (1), although they are written in another form*.

The results of the present work are easily extended to diffusion processes and to more general classes of processes for which Martin boundary theory has been constructed. The only essential restriction is the above-mentioned requirement F. From this point of view, the results of ([2]), where general boundary conditions are described for certain diffusion processes, receive a new interpretation.

* Feller' s investigations were continued in the works of Williams ([5]) and Chung Kai-lai ([6]). The subject of these works is the same classes of Markov processes as in the present note. The general form of the corresponding resolvents is found in terms of certain characteristics, the probabilistic meaning of which is studied in detail in ([6]). However, these characteristics are rather complicated, and the question of how simply one can prescribe their values remains open.

2. Consider a Markov process with transition densities $a(x, y)$, which is killed at the moment ζ of the first accumulation of jumps. Denote by $p(t, x, y)$ its transition function and put

$$g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y)\, dt; \qquad g(x, y) = g_0(x, y).$$

The points x for which $P_x\{\zeta = \infty\} = 1$ are of little interest in the case of our problem and can be eliminated. Therefore, following Feller, we introduce

Assumption C. $P_x\{\zeta < \infty\} > 0$ for all $x \in E$.

Under Assumption C it is proved that: a) $g(x, y) < \infty$ for all $x, y \in E$; b) if $\xi(x) > 0$ $(x \in E)$ and $\sum_x \xi(x) < \infty$, then the values $\Xi(y) = G^*\xi(y)$ are finite and positive for all $y \in E$; c) if $1 \geq l(x) > 0$ $(x \in E)$ and $\sum_y g(y, y)l(y) < \infty$, then the function $L = Gl$ is everywhere finite and positive*.

Associate with each point $z \in E$ the function $k_z(x) = g(x, z)/\Xi(z)$ and the measure $\varkappa_z(y) = g(z, y)/L(z)$. One may embed the space E in a compact set $E \cup B$ and associate with each point $a \in B$ a function $k_a(x)$ so that $k_a \neq k_{\tilde{a}}$ for $a \neq \tilde{a}$ and $k_z(x) \to k_a(x)$ as $z \to a$. Similarly, one may embed the space E in a compact set $E \cup \widehat{B}$ and associate with each point $\rho \in \widehat{B}$ a measure $\varkappa_\rho$ so that $\varkappa_\rho \neq \varkappa_{\tilde{\rho}}$ for $\rho \neq \tilde{\rho}$ and $\varkappa_z(y) \to \varkappa_\rho(y)$ as $z \to \rho$.

Denote by H^λ the cone of all nonnegative solutions of the equation $\mathfrak{A}f = \lambda f$, and by $\widehat{H}^\lambda$ the cone of all nonnegative solutions of the equation $\mathfrak{A}^*\eta = \lambda\eta$. It is proved that for any $\lambda > 0$, $k_a^\lambda = k_a - \lambda G_\lambda k_a \in H^\lambda$, $\varkappa_\rho^\lambda = \varkappa_\rho - \lambda G_\lambda^* \varkappa_\rho \in \widehat{H}^\lambda$. Assign a point $a \in B$ to the set B_0 if: a) k_a is an extreme element of the cone H^0; **b)** $k_a^\lambda \neq 0$ **for all** $\lambda > 0$; **c) the function** k_a **is bounded. Assign a point** $\rho \in \widehat{B}$ **to the set** $\widehat{B}_0$ **if: a)** $\varkappa_\rho$ **is an extreme element of the cone** $\widehat{H}^0$; **b)** $\varkappa_\rho^\lambda \neq 0$ **for all** $\lambda > 0$; **c)** $(1, \varkappa_\rho^\lambda) < \infty$ **for all** $\lambda > 0$*. It is proved that under conditions F and C, B_0 is finite and every bounded function f from H^λ

is uniquely represented in the form of a linear combination of the functions k_a^λ $(a \in B_0)$, while every finite measure $\eta \in \widehat{H}^\lambda$ is uniquely written in the form

$$\eta(y) = \int_{\widehat{B}_0} \varkappa_\rho^\lambda(y)\, \mu(d\rho),$$

where μ is a finite measure on $\widehat{B}_0$ (λ is a positive constant).

Let $f(x)$ be a function on the space E, and let $a \in B_0$. We shall agree to denote by $f(a)$ the fine limit of $f(x)$ as $x \to a$**** (it coincides with the limit in the topology of the compact set $E \cup B$, if the latter limit exists). Under our assumptions, $k_a^\lambda(a) > 0$, and we may put

$$p_a^\lambda(x) = k_a^\lambda(x)/k_a^\lambda(a).$$

It is proved that every function $F \in \mathcal{D}$ is uniquely represented (for any $\lambda > 0$) in the form

$$F = G_\lambda f + \sum_{a \in B_0} F(a) p_a^\lambda.$$

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* The operators G_λ and G_λ^* are constructed from the kernel $g_\lambda(x, y)$ in the same way as the operators $\mathfrak{A}$ and $\mathfrak{A}^*$ from the kernel $a(x, y)$.

** An element f of the cone H is called **extreme** if from the decomposition $f = f_1 + f_2$ $(f_1, f_2 \in H)$ it follows that $f_1 = c_1 f$, $f_2 = c_2 f$, where c_1 and c_2 are numbers.

*** By (f, η) is denoted the sum $\sum_{x \in E} f(x)\eta(x)$.

**** For the definition of fine limit see, for example, ([3], [4]).

3. Put

$$[F]_\rho = \lim_{\lambda \to +\infty} (F, \lambda \varkappa_\rho^\lambda).$$

Obviously,

$$[F]_\rho = \lim_{\lambda \to +\infty} \left(\frac{F}{L}, \mu_\rho^\lambda \right),$$

where $\mu_\rho^\lambda(y) = L(y)\lambda\varkappa_\rho^\lambda(y)$. It can be shown that as $\lambda \to \infty$ the measures μ_ρ^λ converge weakly to the unit measure concentrated at the point ρ. Therefore, if the ratio $F(x)/L(x)$ is bounded and has a limit as $x \to \rho$ (in the topology

$E \cup \widehat{B}_0$), then this limit coincides with $[F]_\rho$. It is natural to regard $[F]_\rho$ as an analogue of the normal derivative at the boundary point ρ.

Let μ be a measure on $\widehat{B}_0$. Put $[F]_\mu = \int_{\widehat{B}0} [F]_\rho \mu(d\rho)$, under the assumption that $[F]_\rho$ is defined for almost all ρ and the integral is meaningful.

Let F be a function on E, and let ν be a measure on $E \cup B_0$. Put

$$\{F, \nu\} = (F, \nu) + \sum_{\alpha \in B_0} F(\alpha)\nu(\alpha)$$

under the assumption that the thin limits $F(\alpha)$ exist for all $\alpha \in B_0$ and (F, ν) is meaningful.

Let an arbitrary partition of the set B_0 into classes be given, and let to each $\alpha \in B_0$ there correspond a nonnegative constant c_α, a measure b_α on $\widehat{B}_0$, and a measure ν_α on $E \cup B_0$, with the following conditions satisfied:

A. $c_\alpha + b_\alpha(\widehat{B}_0) + \nu_\alpha(E \cup B_0) > 0$.

B. If α and β belong to the same class, then $c_\alpha = c_\beta$, $b_\alpha = b_\beta$, $\nu_\alpha = \nu_\beta$, and $\nu_\alpha(\beta) = 0$.

C. Let $p^\lambda_{\bar{\alpha}}$ denote the sum of p^λ_β over all β belonging to the same class as α. For any $\lambda > 0$ the values $\{1 - p^\lambda_{\bar{\alpha}}, \nu_\alpha\}$ and $[1 - p^\lambda_{\bar{\alpha}}]_{b_\alpha}$ are defined and finite.

D. If $b_\alpha(\widehat{B}_0) = b_\beta(\widehat{B}_0) = 0$ and $\nu_\alpha(E) < \infty$, $\nu_\beta(E) < \infty$, then $\nu_\alpha(\beta) = 0$.

We shall denote by $\bar{B}_0$ the collection of classes into which B_0 is partitioned. We shall say that a function $F(z)$ $(z \in E)$ satisfies the boundary condition $Y(\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha)$ if: a) the thin boundary values $F(\alpha)$ $(\alpha \in B_0)$ exist and are constant on each class from $\bar{B}_0$; b) the values $\{F - F(\alpha), \nu_\alpha\}$ and $[F - F(\alpha)]_{b_\alpha}$ are defined and finite for all $\alpha \in B_0$;

c)

$$-c_\alpha F(\alpha) + \{F - F(\alpha), \nu_\alpha\} + [F - F(\alpha)]_{b_\alpha} = 0 \quad (\alpha \in B_0). \tag{1}$$

We shall say that a semigroup of operators in the space C is described by the boundary condition $Y(\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha)$ if the domain of definition of its weak infinitesimal operator A consists of all functions $F \in \mathcal{D}$ satisfying the condition $Y(\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha)$.

The solution of the problems posed in § 1 is given by the following theorem.

Theorem 1. *Every $\mathfrak{A}$-semigroup is described by some boundary condition $Y(\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha)$. An arbitrary boundary condition $Y(\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha)$ describes some $\mathfrak{A}$-semigroup.*

4. The resolvent of an $\mathfrak{A}$-semigroup can be expressed by explicit formulas in terms of the additional characteristics $\{\bar{B}_0, c_\alpha, \nu_\alpha, b_\alpha\}$. Denote by $\bar{\alpha}$ the class from $\bar{B}_0$ containing the point α, and put

$$c_{\bar{\alpha}} = c_\alpha, \quad \nu_{\bar{\alpha}} = \nu_\alpha, \quad b_{\bar{\alpha}} = b_\alpha, \quad F(\bar{\alpha}) = F(\alpha), \quad p^\lambda_{\bar{\alpha}} = \sum_{\beta \in \bar{\alpha}} p^\lambda_\beta.$$

Introduce the abbreviated notation

$$\mathfrak{R}_{\bar{a}}(F) = \{F, \mathbf{v}_{\bar{a}}\} + [F]_{b_{\bar{a}}}$$

and define the matrix $a^\lambda_{\bar{\alpha}\bar{\beta}}$ $(\bar{\alpha}, \bar{\beta} \in \widehat{B}_0)$ by the formulas $a^\lambda_{\bar{a}\bar{\beta}} = -\mathfrak{R}_{\bar{a}}(p^\lambda_{\bar{\beta}})$ for $\bar{a} \neq \bar{\beta}$, $a^\lambda_{\bar{a}\bar{a}} = c_{\bar{a}} + \mathfrak{R}_{\bar{a}}(1 - p^\lambda_{\bar{a}})$.

Theorem 2. *For every $\lambda > 0$ the matrix $a^\lambda_{\bar{\alpha}\bar{\beta}}$ has an inverse matrix $\Gamma^\lambda_{\bar{\alpha}\bar{\beta}}$ with nonnegative elements. The resolvent of the $\mathfrak{A}$-semigroup is given by the formula*

$$R_\lambda f = G_\lambda f + \sum_{\bar{a} \in \bar{B}_0} \left\{ (G_\lambda f, N^\lambda_{\bar{a}}) + [G_\lambda f]_{M^\lambda_{\bar{a}}} \right\} p^\lambda_{\bar{a}},$$

where

$$N^\lambda_{\bar{a}} = \sum_{\bar{\beta} \in \bar{B}_0} \Gamma^\lambda_{\bar{a}\bar{\beta}} \mathbf{v}_{\bar{\beta}}, \qquad M^\lambda_{\bar{a}} = \sum_{\bar{\beta}} \Gamma^\lambda_{\bar{a}\bar{\beta}} \mathbf{b}_{\bar{\beta}}.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.