ZERO-SETS AND A HOMEOMORPHISM WITH FINITE DIRICHLET INTEGRAL

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Abstract

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ZERO-SETS AND A HOMEOMORPHISM WITH FINITE DIRICHLET INTEGRAL

(Presented by Academician M. A. Lavrent' ev, 15 VI 1966)

One says that a mapping $T(z) = u(x, y) + iv(x, y)$ of a domain G of the plane R^2 belongs to the class $BL(G)$ if the functions u and v are continuous, have generalized first derivatives with respect to x and y in the domain G, and have finite Dirichlet integral

$$D(G) \stackrel{\text{def}}{=} \iint_G \left(|\nabla u|^2 + |\nabla v|^2\right) dx\, dy.$$

It is known that every KQC-mapping (K-quasiconformal mapping) of a domain G onto a domain Δ of finite area belongs to $BL(G)$.

Let G be a simply connected domain, $\dot{G} = \overline{G} \setminus G$ the boundary of G, and $\widetilde{G}$ a compact extension of G, whose elements are the prime ends in the sense of Carathéodory (if G is unbounded, the corresponding construction using the spherical metric on the plane is understood) (see ([1–3])).

In this note we give a condition for the invariance of zero-sets $\subset \dot{G}$ with respect to homeomorphisms of class BL, in particular, the conditional boundary N-property of BL-homeomorphisms of G onto a domain Δ with rectifiable boundary.

Our condition (see Definition 2 below) resembles the metric criteria for the null boundary of Riemann surfaces:

$$\int^{\rho} \frac{dt}{\nu(M_t)}$$

diverges (see ([4]), pp. 408–418). This analogy is not merely external.

I. A homeomorph $\gamma \subset G$ of an open or half-open interval I (of the number line) is called a section of G if $G \setminus \gamma$ consists of exactly two components.

Let G_a^{γ} be the component of the set $G \setminus \gamma$ containing $a \in G$, and let $U \subset G$ be some closed disk with center at the point a.

By definition, a prime end $e \in \dot{G}$ is covered by the section γ if e belongs to $G \setminus \overline{G_a^\gamma}$ and $U \subset G_a^\gamma$; a set $E \subset \dot{G}$ is covered by a system of sections $\{\gamma\}$ if every element $e \in E$ is covered by at least one $\gamma \in \{\gamma\}$.

Definition 1 (cf. ([5])). A set $E \subset \dot{G}$ is called a 0-set (a generalized null-set) if, for every $\varepsilon > 0$, there exists a countable system $\{\gamma_i\}$ of sections of G covering E, with

$$\sum_i l(\gamma_i) < \varepsilon,$$

where l is the length of γ_i.

For each $e \in E \subset \dot{G}$, let Z^e be some nonempty set of principal points e (([3]), p. 194). Put

$$M \overset{\text{def}}{=} M(E) = \bigcup_{e \in E} Z^e, \qquad L_t \overset{\text{def}}{=} \{z \in R^2 : \ r(z, M) = t\} \quad \text{and} \quad M_t \overset{\text{def}}{=} L_t \cap G;$$

here r is Euclidean distance.

Definition 2. A set $E \subset \dot{G}$ is called a 0_1-set if there exists such a bounded set $M(E) = M$ that $r(M, U) =$

$= \rho > 0$ and

$$\int_0^\rho \frac{dt}{\nu(M_t)}$$

diverges (ν is Hausdorff length, see ([6]), p. 92).

The condition $\rho > 0$ is, obviously, equivalent to the nonemptiness of $\dot{G} \setminus \widetilde{E}$, where $\widetilde{E}$ is the closure of E in G.

It is proved that an $\dot{\theta}_1$-set is an $\dot{\theta}$-set.

It is known that a BL-homeomorphism $T : G \to \Delta$ can be extended to a continuous mapping of $\dot{G}$ onto Δ (([3]), p. 66).

Let $l[T(M_t)]$ be the sum for the images of the component arcs of the set M_t (see ([7])). With the aid of the inequality (see ([8]), p. 14)

$$\int_0^\rho \frac{l^2[T(M_t)]}{\nu(M_t)}\, dt \leqslant D(G)$$

one proves

Theorem 1*. A homeomorphism $T \in BL(G)$ of a simply connected domain G onto a domain Δ takes every $\dot{\theta}_1$-set into a $\dot{\theta}$-set $T(E) \subset \Delta$. If E is closed in $\dot{G}$, and $\nu(\Delta) < \infty$ (here Δ is a continuous image of the circle; see ([10]), p. 51, $\overline{u}$ ([11]), pp. 411–412), then $\nu(|T(E)|) = 0$, and therefore (see ([12])) $|T(E)|$ is a 0-set in the sense of Painlevé ($|T(E)|$ is the sum of the bodies of all $e \in T(E)$).

If G is homeomorphic to a circle, then the sets E and $M(E)$ can be identified (see ([2]), p. 408).

As is known ([13]), there exists a KQC-mapping $T(z)$ of the disk $G' : |z| < 1$ onto the disk $\Delta' : |w| < 1$ such that for some perfect set $E \subset G'$, with $\nu(E) = 0$, one has $\nu(|T(E)|) > 0$. Hence we have

Corollary 1. There exists a perfect set of measure zero on the boundary of the disk (and hence also a 0-set in the sense of Painlevé) which is not an $\dot{\theta}_1$-set. Moreover, there exists on the x-axis in the plane R^2 a perfect set $P \subset [0, 2\pi]$ of measure zero for which

$$\int_0^\rho \frac{dt}{\nu(P_t)}$$

converges for every $\rho \in (0, \infty)$ and, consequently,

$$\lim_{t \to 0} \nu(P_t)/t = \infty;$$

here

$$R_t = \{z \in R^2 : r(z, P) = t\}.$$

II. A homeomorph $p \subset G$ of the half-open interval $I = [0, 1)$ is called a **path**. Let $p = f(I)$, and $p_\alpha = f(I_\alpha)$, where f is a homeomorphism, and $I_\alpha = (\alpha, 1)$, $\alpha \in (0, 1)$. One says that the path p **goes to a prime end** ($p \to e$) if, for every open set g containing the prime end e, there exists an α such that $f(I_\alpha) \subset g$ (see ([14]), p. 5).

Let $|e|$ be the body (the limiting set) of the prime end $e \in \dot{\Delta}$,

$$\Delta_w = \{e \in \dot{\Delta} : w \in |e|\}$$

and

$$\Phi^w = \{e \in \Delta_w : w \text{ is the principal point of } e\}.$$

Definition 3. A homeomorphism $T : G \to \Delta$ **concentrates** a set $E \subset \dot{G}$ if there exists a point $w \in \overline{\Delta}$ with the following property: for each $e \in E$ there is a path $p_e \to e$ such that $T(p_e) \to \varphi \in \Phi^w$.

Theorem 2. If $E \subset \dot{G}$ is not a $\dot{\theta}$-set, then there does not exist a homeomorphism $T : G \to \Delta$, $T^{-1} \in BL(\Delta)$, concentrating E (i.e., it concentrates only sufficiently "sparse" sets).

Corollary 2. Let G be a simple rectifiable closed arc. If $E \subset G$ and $\nu(E) > 0$, then a homeomorphism $T : G \to \Delta$, $T^{-1} \in BL(\Delta)$, does not concentrate E; in particular, there does not exist a homeomorphism T, $T^{-1} \in BL(\Delta)$, taking one and the same angular boundary value w on a set E (for conformal mappings the latter is, of course, a consequence of the well-known theorem of Luzin-Privalov, ([9]), p. 292).

* It is interesting to compare this assertion with Theorems 3 and 4 of ([5]), and also with the theorem of M. A. Lavrent' ev; see ([9]), p. 293.

We shall state one theorem concerning the topological structure of the set $\dot{\Delta}_w$.

Theorem 3. *The set $\dot{\Delta}_w$ is closed in $\dot{\Delta}$.*

Remark 1. The assertions are easily extended to finitely connected domains.

Remark 2. In many respects analogous results can be obtained for mappings of spatial domains.

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Note: Figure translations are in progress. See original paper for figures.