EXTENSION OF FUNCTORS FROM THE CATEGORY OF BANACH SPACES TO THE CATEGORY OF \((\alpha)\)-SPACES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.83 + 513.88 + 519.50

MATHEMATICS

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EXTENSION OF FUNCTORS FROM THE CATEGORY OF BANACH SPACES TO THE CATEGORY OF (α)-SPACES

(Presented by Academician P. S. Novikov on 11 VI 1966)

1°. Let E be a separable locally convex space and A a bounded absolutely convex set (a ball) in E. By E^A we shall denote the linear hull of A, endowed with the norm

$$\|x\|_A = \inf\{\rho > 0 : \ x \in \rho A\}.$$

The canonical embedding $i^A : E^A \to E$ is continuous.

If E^A is a Banach space, then A is called a **Banach ball** (1), and E^A a Banach subspace in E. The space E is called **locally complete** (1) if every sequence of its elements contained in some E^A and forming there a Cauchy sequence converges in E. A separable locally convex space E is called a **space of type** (α), or an (α)-space (1), if it is locally complete and is a (β)-space (4), i.e. representable as an inductive limit of Banach spaces.

The class of all (α)-spaces together with all continuous linear mappings of each of them into each forms a category, which we shall denote by $\mathfrak{A}$. The class $\mathfrak{B}$ of all Banach spaces constitutes a full subcategory in $\mathfrak{A}$.

In the present note we prove the possibility of extending any functor in the category $\mathfrak{B}$ (or defined on $\mathfrak{B}$ and taking values in $\mathfrak{A}$) to a functor in the category $\mathfrak{A}$. Such an extension is, generally speaking, non-unique. The existence of two "extreme" extensions is established. As a consequence, in particular, one obtains a certain description of the totality of functors in $\mathfrak{A}$ whose restrictions to $\mathfrak{B}$ coincide.

2°. Spaces of type (α) were first singled out and studied by D. A. Raikov in (1). We recall some facts and consequences from that work.

Proposition 1. *Let $\{A\}$ be a complete system, i.e. the system of all Banach balls of the (α)-space E. Then E is the limit of the direct spectrum of its Banach subspaces $\{E^A\}$ with respect to the embeddings*

$$i^A : E^A \to E.$$

Every separable locally convex space can be locally completed (¹). For spaces of countable character, in particular for normed spaces, local completion coincides with ordinary completion.

The topology of any separable locally convex space E can be (β)-strengthened, i.e. can be strengthened so that, in the new topology, E becomes a (β)-space with the same stock of Banach balls.

Local completeness is stable with respect to (β)-strengthening of the topology. A locally completed (α)-space is again an (α)-space.

Let E and F be locally convex spaces. By $L(E,F)$ we shall denote the vector space of all continuous linear mappings of E into F. The space $L(E,F)$ in the topology of simple convergence will be denoted by $L_s(E,F)$, and its (β)-strengthening by $L_i(E,F)$. The topology of the space $L_i(E,F)$ is called inductive (¹). From Theorems 4.4 and 4.8 of (¹) it follows immediately ···

Proposition 2. *If E is a space of type (α), and F is locally complete, then $L_i(E,F)$ is an (α)-space.*

In the category of separable locally convex spaces an inverse spectrum always has a limit. By the limit of a direct spectrum in this same category one understands the separable space associated with the limit of this same spectrum in the category of all locally convex spaces.

Proposition 3. *The limit of a direct (respectively, inverse) spectrum in the category $\mathfrak{A}$ is the locally completed (respectively, (β)-strengthened) limit of the same spectrum in the category of separable locally convex spaces.*

Let $\{A\}$ and $\{B\}$ be complete systems of Banach disks in (α)-spaces E and F, respectively. The tensor products $E^A\widehat{\otimes}F^B$ (⁴) form a direct spectrum of Banach spaces. The limit of this direct spectrum in the category $\mathfrak{A}$ is called the tensor (α)-product of the spaces E and F (¹) and is denoted by $E\widehat{\otimes}F$. The space $E\widehat{\otimes}F$, obviously, is an (α)-space.

3^0. In the definition of a (covariant) functor $S : \mathfrak{C} \to \mathfrak{A}$ ($\mathfrak{C}$ is a subcategory of $\mathfrak{A}$), in addition to the known conditions, we shall assume that the mapping $\varphi \to S\varphi$ from the space $L_i(E,F)$ into the space $L_i(SE,SF)$, generated by the functor S, is linear and continuous.

Basic functors. Let E be a fixed space of type (α). The correspondence $F \to L_i(E,F)$, where $F \in \mathfrak{A}$, defines a functor (Proposition 2) in the category $\mathfrak{A}$, which will be denoted by Ω_E. On morphisms Ω_E is defined in the natural way. The tensor (α)-product defines a functor Σ_E as follows: $\Sigma_{EF} = E\widehat{\otimes}F$.

It was established in (¹) that the category $\mathfrak{A}$ with the functors Σ_E and Ω_E forms a D-category in the sense of A. S. Schwarz (²). Since the functor Σ_E is left adjoint to the functor Ω_E (and Ω_F to Σ_E is right) in the sense of Kan (⁶), it follows that

Proposition 4 (2). *The functor Σ_E (respectively, Ω_E) is permutable with the limits of direct (respectively, inverse) spectra in the category $\mathfrak{A}$.*

4^0. Consider a functor $T : \mathfrak{B} \to \mathfrak{B}$ (or $T : \mathfrak{B} \to \mathfrak{A}$). The collection P_T of all its extensions to a functor in $\mathfrak{A}$ forms a category if, as $\mathrm{Hom}(T^\gamma, T^\delta)$ ($T^\gamma, T^\delta \in P_T$), one takes all those mappings of T^γ into T^δ whose restrictions to $\mathfrak{B}$ coincide with the identity mapping of the functor T into itself.

Main theorem. *Every functor $T : \mathfrak{B} \to \mathfrak{B}$ (or $T : \mathfrak{B} \to \mathfrak{A}$) admits an extension to a functor in $\mathfrak{A}$, and in the category P_T of all its extensions there is an initial object T^i and a final object T^π, i.e. for any $T^\gamma \in P_T$ there exists a unique mapping $\sigma^{i,\gamma} : T^i \to T^\gamma$ and a unique mapping $\sigma_{\gamma,\pi} : T^\gamma \to T^\pi$.*

The functor T^i is constructed as follows. Let E be an (α)-space and let $\{E^A\}$ be the direct spectrum of its Banach subspaces determined by a complete system of Banach disks $\{A\}$ in E (Proposition 1). By T^{iE} we denote the limit of the corresponding direct spectrum $\{TE^A\}$ in $\mathfrak{A}$ (Proposition 3).

Let E be a separable locally convex space and let $\{U\}$ be a fundamental system of its closed absolutely convex neighborhoods of zero. By E_U we shall mean the quotient space E/N_U (N_U is the largest subspace contained in U), endowed with the usual normed topology. Denote the completion of E_U by $\widehat{E}_U$. The system of Banach spaces $\{\widehat{E}_U\}$ forms an inverse spectrum, whose limit in the category of locally convex spaces coincides with the completion $\widehat{E}$ of the space E (5). If E is a space of type (α), then the space $T^\pi E$ is defined as the limit of the inverse spectrum $\{T\widehat{E}_U\}$ in the category $\mathfrak{A}$ (Proposition 3).

Denote by $F(\mathfrak{A})$ the category of all functors acting in $\mathfrak{A}$. Let $S \in F(\mathfrak{A})$, and let $S|_{\mathfrak{B}}$ be the restriction of S to $\mathfrak{B}$. From the main theorem it follows immediately:

Proposition 5. *The category $P_{S|_{\mathfrak{B}}}$ forms a spectrum* (2) *in $F(\mathfrak{A})$, whose projective limit is the initial object S^i, and whose inductive limit is the final object S^π.*

5°. Proposition 6. *If the functor $S : \mathfrak{A} \to \mathfrak{A}$ is permutable with limits of direct spectra in $\mathfrak{A}$, then $S^i = S$.*

Proposition 7. *If the functor $S : \mathfrak{A} \to \mathfrak{A}$ is permutable with limits of inverse spectra in $\mathfrak{A}$, then S and S^π coincide on complete spaces of the category $\mathfrak{A}$.*

Example 1. Denote by I the identity functor in the category $\mathfrak{A}$. From Proposition 6 it follows that $I^i = I$. If E is a complete (α)-space, then (Proposition 7) $I^\pi E = E$. In the general case, to obtain $I^\pi E$ one must complete E and then (β)-strengthen its topology.

Example 2. From Propositions 5 and 6 it follows that $\Sigma_E^i = \Sigma_E$.

Example 3. For the functor Ω_E^π the equality

$$\Omega_E^\pi(F) = L_i(E, \hat{F})$$

holds, where $\hat{F}$ is the completion of the (α)-space F. In particular, if F is complete, then (Propositions 5 and 7) $\Omega_E^{\pi}(F) = L_i(E, F)$. The set of all mappings $\varphi \in L(E, F)$ that transform some neighborhood of zero in E into a bounded set in F coincides with the union of the spaces $L_i(E, F^B)$ ($\{B\}$ is a complete system of Banach disks in F). The limit of the direct spectrum $\{L_i(E, F^B)\}$ in the category $\mathfrak{A}$ coincides with the space $\Omega_E^i(F)$. In particular, if E is a Banach space, then $\Omega_E^i = \Omega_E$.

Example 4. Each space of numerical sequences l_p $(1 \leq p \leq \infty)$ defines in the category $\mathfrak{B}$ the corresponding functor $\bar{l}_p$ (3). Let E be a space of type (α). Consider the space of all sequences (x_n) from the completion $\hat{E}$ for which the series

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges absolutely in $\hat{E}$, whatever numerical sequence $(\alpha_n) \in l_q$, $p^{-1} + q^{-1} = 1$. As a set of elements this space coincides with $\bar{l}_p^{\pi}(E)$. The set of all sequences (x_n) from E for which the series

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges absolutely in some fixed Banach subspace of E, whatever numerical sequence $(\alpha_n) \in l_q$, $p^{-1} + q^{-1} = 1$, is naturally turned into a Banach space. The limit of the direct spectrum in the category $\mathfrak{A}$ of Banach spaces obtained in this way coincides with $\bar{l}_p^i(E)$.

In conclusion, the author expresses his deep gratitude to Prof. D. A. Raikov for his attention to the work, very valuable discussions, and comments.

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Received
26 V 1966

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