INTERPOLATION OF LINEAR OPERATORS AND ESTIMATES OF FOURIER COEFFICIENTS

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Abstract

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MATHEMATICS

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INTERPOLATION OF LINEAR OPERATORS AND ESTIMATES OF FOURIER COEFFICIENTS

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In the first section of the present paper new interpolation theorems are proved, and in the second section these theorems, as well as known ones, are applied to estimates of the Fourier coefficients of functions belonging to various Banach spaces.

We shall denote the Fourier coefficients of a summable function $x(t)$ with respect to an orthonormal system $f_k(t)$ $(k = 1, 2, ...)$ by $c = \{c_k\}$, and the rearrangement of their moduli in decreasing order by $c^* = \{c_k^*\}$. The expansion of the function $x(t)$ in a Fourier series gives rise to the family of operators

$$J_s x = \{k^s c_k\} \qquad (s \geqslant 0),$$

acting from function spaces into sequence spaces.

1°. We shall call a Banach space E of measurable functions **symmetric** if the norm of the space E is monotone and invariant under measure-preserving transformations of the argument (^{1}). A symmetric space gives rise to the so-called fundamental function $\varphi(\tau) = \|\chi_{[0,\tau]}\|_E$.

Let $1 \leqslant \mu \leqslant \nu \leqslant 2$. In the set of summable functions define the subset $Q(\mu, \nu)$ of those $x(t)$ for which

$$\mu \int_0^\tau x^*(t)\,dt \leqslant \int_0^{2\tau} x^*(t)\,dt \leqslant \nu \int_0^\tau x^*(t)\,dt$$

for all $\tau \in [0, 1/2]$, where $x^*(t)$ is the rearrangement of the function $|x(t)|$ in decreasing order.

The **associate space** E' is the set of functionals from E^* that are representable in the form of Lebesgue integrals, with norm E^* (^{2}). In what follows we shall assume that E is separable or is associated with a separable space.

Theorem 1. Let E be a symmetric space. If

$$\mu < \underline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \nu,$$

then

$$\|x\|_1 = \sup_{\substack{\|y\|_{E'}=1 \\ y\in Q(\mu,\nu)}} \int_0^1 x(t)y(t)\,dt$$

is equivalent to the original norm of the space E.

Theorem 1 is very essential in the proof of Theorems 2, 3, 4, 6, 7, 10.

Theorem 2. Let a linear operator A be continuous in each space $\mathcal{L}_p$ $(1 \leqslant p_0 < p < p_1 \leqslant \infty)$. Then A is continuous in E, if the fundamental function satisfies the inequalities

$$\sqrt[p_1]{2} < \underline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \sqrt[p_0]{2}.$$

This theorem generalizes the results of (^{3}).

For the formulation of a more general result we shall need the following construction. If E is a symmetric space and $\beta > 0$, then $E_\beta = \{x : x^*(t)t^{-\beta} \in E\}$ and

$$\|x\|_{E_\beta} = \|x^* t^{-\beta}\|_E.$$

Clearly, in order that E_β be nonempty, it is necessary and sufficient that $t^{-\beta} \in E$. In this case E_β is a symmetric space.

Theorem 3. Let the linear operator A be continuous from the space $\mathcal{L}_{1/\alpha}$ into $\mathcal{L}_{1/(\alpha-\Delta)}$ for each $\alpha \in (\alpha_0, \alpha_1)$, where $0 < \Delta \leqslant \alpha_0 \leqslant \alpha_1 \leqslant 1$. Then A is continuous from E into E_Δ, if the fundamental function of the space E_Δ satisfies the inequalities

$$2^{\alpha_0-\Delta} < \lim_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < 2^{\alpha_1-\Delta}.$$

Remark 1. When $E = \mathcal{L}_p$, this proposition in a more general form has recently been obtained by V. A. Dikarev and V. I. Matsaev (^{4}). It is interesting to note that in this case Theorem 3 asserts more than M. Riesz' s interpolation theorem.

Remark 2. Theorems 1-3 carry over verbatim to symmetric spaces of sequences. In this case the role of the fundamental function is played by the fundamental sequence

$$\varphi(n) = \| \underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots \|_E.$$

Remark 3. Theorems 2 and 3 are valid not only for linear operators, but also for quasilinear ones.

As I. Ts. Gokhberg and M. G. Krein pointed out to the author, an immediate consequence of the analogue of Theorem 2 for sequences, Theorem 11.1 (^{5}), and the results of (^{3,6}) is

Theorem 4. Let σ_Φ be a symmetrically normed ideal of completely continuous operators in a separable Hilbert space H; $\Phi(\xi)$ a symmetric norming function. Let A be a Volterra operator, and let A_R and A_I be its real and imaginary components. In order that

$$\sup_{A_I \in \sigma_\Phi} \frac{\|A_R\|_{\sigma_\Phi}}{\|A_I\|_{\sigma_\Phi}} < \infty,$$

it is necessary and sufficient that, for some $\varepsilon > 0$ and all sufficiently large n, the inequalities

$$(1+\varepsilon)\Phi(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots) \leqslant \Phi(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots) \leqslant$$

$$\leqslant (2-\varepsilon)\Phi(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots)$$

hold.

Let $\psi(t)$ be an increasing concave function, $\psi(0) = 0$. By $\Lambda(\psi)$ $(\lambda(\psi))$ (^{7}) we denote the Banach space of functions (sequences) for which

$$\|x\|_{\Lambda(\psi)(\lambda(\psi))} = \int_0^\infty \psi(n_x(\tau))\, d\tau,$$

where $n_x(\tau) = m\{t : |x(t)| > \tau\}$. In the set of increasing concave functions we define the transformation

$$\bar{\psi}(t) = t\psi(1/t).$$

Theorem 5. If the linear operator A is continuous from $\mathcal{L}_1$ to l_∞ and from $\mathcal{L}_2$ to l_2, then A is continuous from $\Lambda(\psi)$ to $\lambda(\overline{\psi})$, if

$$\lim_{t\to 0}\frac{\psi(2t)}{\psi(t)} > \sqrt{2}.$$

2°. To estimate the Fourier coefficients of functions belonging to various Banach spaces, in (8) the following method is systematically used. A certain interpolation theorem is proved, which is then applied to the operator J_s with a suitable choice of the parameter s. We shall also use this example.

From Theorems 1 and 5 it follows that

Theorem 6. Let the functions $f_k(t)$ be uniformly bounded. If the fundamental function of the space E satisfies the inequalities

$$\sqrt{2} < \lim_{t\to 0}\frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0}\frac{\varphi(2t)}{\varphi(t)} < 2,$$

then

$$\left\|\sum_{k=1}^{\infty} k c_k^* x_{\left[\frac{2\pi}{k+1},\, \frac{2\pi}{k}\right]}\right\|_E \leq L\|x\|_E,$$

where L does not depend on $x(t)$.

In the case where E is the Orlicz space $\mathcal{L}_M^*$ (9), Theorem 6 can be formulated as follows:

If

$$2 < \lim_{u\to\infty}\frac{M(2u)}{M(u)}, \qquad \overline{\lim}_{u\to\infty}\frac{M(2u)}{M(u)} < 4$$

and $x(t) \in \mathcal{L}_M^*$, then

$$\sum_{k=1}^{\infty}\frac{1}{k^2}M(kc_k^*) < \infty.$$

In order to obtain from this proposition Paley' s theorem (8), it is enough to put $M(u) = u^p$ $(1 < p < 2)$.

For the trigonometric system and functions with monotonically decreasing Fourier coefficients, a more precise assertion is valid. Put

$$x(t) = \sum_{k=1}^{\infty} a_k \sin kt.$$

Theorem 7. Let the fundamental function of the space E satisfy the inequalities

$$1 < \lim_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < 2$$

and $a_k \downarrow 0$. In order that $x(t) \in E$, it is necessary and sufficient that

$$\sum_{k=1}^{\infty} k a_k x_{\left[\frac{2\pi}{k+1}, \frac{2\pi}{k}\right]}(t) \in E.$$

Special cases of Theorem 7 are the theorems of Hardy-Littlewood (^{8}), G. G. Lorentz (^{7}), V. A. Dikarev and V. I. Macaev (^{4}).

For what follows we shall need the definitions of the spaces $W_p^s(\Omega)$, $H_p^s(\Omega)$ (^{10}), and $l_{p,\nu}$:

$$\|x\|_{p,\nu} = \left(\sum_{k=1}^{\infty} \frac{1}{k^2} |x_k|^p \right)^{1/p}.$$

Without formulating the two general interpolation theorems, we shall present their concrete realizations for the operator J_s.

With the aid of the theory of scales of Banach spaces ($^{11}, ^{12}$) one proves:

Theorem 8. *Let $f_k(t)$ be the trigonometric system. If $\alpha > 2/p - 1,\ 1 < p < 2$, then the operator J_1 is continuous from $W_p^\alpha[0, 2\pi]$ into $l_{2/(1-\alpha),\, v}$.*

Theorem 9. *The operator J_s acts continuously from $H_p^s[0, 2\pi]$ into $l_{p',v}$, if $1 \le p \le 2,\ s \ge 0$.*

We prove Theorem 9 by combining the method of analytic scales of S. G. Krein (^{11}) and the complex method of A. P. Calderón (^{10}). From Theorem 7 one can obtain the well-known theorem of S. N. Bernstein on the absolute convergence of Fourier series.

A direct consequence of (^{13}) and Theorem 2 is

Theorem 10. *If the fundamental function of the space satisfies the inequalities*

$$\sqrt[4]{2} < \lim_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}, \qquad \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)} < \sqrt[4]{8}, \tag{1}$$

then

$$\|S_n x\|_E \le C \|x\|_E,$$

where S_{nx} *is the segment of the Fourier series of the function* $x(t)$ *with respect to the system of Legendre functions, and* C *does not depend on* $x(t)$ *or on* n*.*

We note that the constants $\sqrt[4]{2}$ and $\sqrt[4]{8}$ in (1) are sharp; i.e., the theorem ceases to be true if $\sqrt[4]{2}$ is replaced by $\sqrt[4]{2}-\varepsilon$, or $\sqrt[4]{8}$ by $\sqrt[4]{8}+\varepsilon$, for any $\varepsilon > 0$. The same may be said about the constants in Theorems 2-7.

3°. A comparison of the interpolation theorems of M. Riesz and E. M. Stein–G. Weiss (^{14}) shows the naturalness of the following question: does the boundedness of the linear operator A in $\mathcal{L}_p$ $(p \in (p_0, p_1))$ follow from the inequalities

$$\|A\chi_{[a,b]}\|_{\mathcal{L}_{p_i}} \leq \|\chi_{[a,b]}\|_{\mathcal{L}_{p_i}} \qquad (i = 0, 1)$$

for arbitrary $a, b \in [0, 2\pi]$?

The answer to this question is, generally speaking, negative. It is well known that the operator J_1 is unbounded from $\mathcal{L}_p$ into $l_{p,v}$ $(p > 2)$ (^{8}). However, it is easy to verify that for arbitrary $a, b \in [0, 2\pi]$ the inequalities

$$\|J_1\chi_{[a,b]}\|_{l_{p,v}} < \|\chi_{[a,b]}\|_{\mathcal{L}_p} \qquad (p > 2)$$

hold.

Thus, from the uniform boundedness of an improving linear operator on characteristic functions of all possible intervals in the scale $\mathcal{L}_p$, it does not follow, generally speaking, that A is bounded in $\mathcal{L}_p$.

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