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Abstract

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MATHEMATICS

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FINITE-DIMENSIONAL REPRESENTATIONS OF CERTAIN BICOMPACT TOPOLOGICAL SEMIGROUPS

(Presented by Academician A. I. Mal' tsev on 8 XII 1966)

1. A **representation** of a topological semigroup A in a topological semigroup A' is a continuous homomorphic mapping of A into A'. If A' is a semigroup of continuous linear transformations of a linear topological space, then the representation is called **linear**. If the representation space is finite-dimensional, then the representation is called **finite-dimensional**.

A representation α separates points $x, y \in A$ if $\alpha(x) \neq \alpha(y)$. We shall say that a topological semigroup A admits a sufficient system of finite-dimensional representations if for any $x, y \in A$ there exists a finite-dimensional representation α separating the points x and y.

A finite-dimensional representation α of a semigroup A is called **irreducible** if the semigroup of linear operators $\alpha(A)$ has no nontrivial invariant subspaces.

A nonzero bounded representation χ of a commutative topological semigroup A in the multiplicative semigroup of complex numbers is called a semicharacter of A. A set of semicharacters X is called **point-separating** if it forms a sufficient system of representations of the semigroup A.

2. The existence of a sufficient system of irreducible representations is the fundamental question of any representation theory. In the present work this question is solved for a totally disconnected bicompact topological semigroup and for an inverse completely regular bicompact topological semigroup (completely regular in the algebraic sense (3)).

In the commutative case, conditions have been found that are necessary and sufficient for the set of semicharacters to separate points. This makes it possible to strengthen Austin' s duality theorem $(1, 4)$.

3. We begin with the study of a totally disconnected bicompact topological semigroup. We shall show that such a semigroup admits a sufficient system of irreducible finite-dimensional representations. In fact, an even stronger result holds.

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Theorem 1. *A totally disconnected bicompact topological semigroup admits a sufficient system of representations in the class of finite semigroups.*

Indeed, without loss of generality one may assume that the totally disconnected bicompact topological semigroup A contains an identity. Let a and b be unequal elements of A, and let P be an open-and-closed neighborhood of the point a not containing b. On A introduce the binary relation ε: $c\varepsilon c'$ if and only if xcy and $xc'y$ simultaneously belong or do not belong to P for all $x, y \in A$.

By direct verification it is not hard to see that the relation ε is a two-sided stable equivalence, and a is not εb. Further, one can show that every equivalence class of ε is an open-and-closed set; hence from the bicompactness of A follows the finiteness

numbers of these classes. Therefore the canonical mapping of A onto A/ε is a representation of A on a finite semigroup and separates the points a and b.

From this theorem it is easy to obtain

Corollary. *The set of quasi-characters of a bicompact totally disconnected commutative semigroup of idempotents separates its points.*

4. Let us recall some properties of a commutative semigroup of idempotents E, which will be useful below (see ([3],[4],[6])). The semigroup E can be ordered by putting $e_\alpha \leq e_\beta$ if and only if $e_\alpha e_\beta = e_\alpha$. With respect to this ordering, E forms a semilattice under intersection. If the commutative semigroup of idempotents is also bicompact, then this semilattice contains a least element.

Lemma 1. *If a bicompact commutative semigroup of idempotents admits a sufficient system of finite-dimensional representations, then it is totally disconnected.*

For the proof it is enough to establish that a commutative bicompact semigroup of idempotent linear operators on an n-dimensional vector space is always finite, since if two distinct elements of a topological semigroup belong to one and the same component, then any n-dimensional representation separating them would have to map this component onto an infinite set of linear operators. It remains to note that, first, every chain in the ordered set of commutative idempotent operators is finite; second, for each operator there exist no more than n elements immediately following it; and, finally, an ordered bicompact commutative semigroup of idempotent operators contains a least element.

From this lemma it is obvious that

Corollary. *In order that a bicompact inverse semigroup admit a sufficient system of finite-dimensional representations, it is necessary that it be totally disconnected.*

5. Theorem 2. *Let A be a bicompact totally regular inverse semigroup, and let E be the subsemigroup of its idempotents. In order that there exist a sufficient*

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system of irreducible finite-dimensional representations of the semigroup A, it is necessary and sufficient that the space E be totally disconnected.

Necessity is a consequence of Lemma 1.

Sufficiency. The semigroup A is a commutative bundle of groups ([3]). Let e_x be the identity of the group which contains the element $x \in A$. The mapping $\lambda(x) = e_x$ of the semigroup A onto E is continuous ([2]) and homomorphic. If the elements $a, b \in A$ belong to different subgroups, then λ separates the points a and b, and it remains to refer to the corollary of Theorem 1. If, however, a, b belong to one and the same subgroup, then there exists an open-and-closed subsemigroup $S \subset E$ such that $e_a = e_b \in S$ and $as \neq bs$ for every $s \in S$. The semigroup S contains a least element f; denote by α_f the shift of the semigroup A induced by it. The mapping α_f is a representation of A on fA, and the image of the open-and-closed subsemigroup $B = \lambda^{-1}(S) \subset A$ is the open-and-closed subgroup fB. Let ν be an irreducible finite-dimensional representation of the bicompact group fB separating the points a and b ([5]). Extend ν to the open-and-closed ideal $fA \setminus fB$ of the semigroup fA by setting $\nu(fA \setminus fB) = 0$, where 0 is the zero operator of the representation space ν. Then $\nu\alpha_f$ is an irreducible finite-dimensional representation of the semigroup A, separating the points a and b.

Directly from the proof of the theorem we obtain

Corollary. *Let a and b be distinct elements of a bicompact totally regular inverse semigroup A, the subsemigroup of idempotents of which is totally disconnected. Then there exists an irreducible finite-dimensional representation of the semigroup A on a group with an adjoined zero, separating the points a and b.*

6. From the corollaries to Theorems 1 and 2 and from the theory of characters of bicompact commutative groups [5] it follows that

Theorem 3. *Let A be a bicompact commutative inverse semigroup; let E be the subsemigroup of idempotents of A; let X be the set of semicharacters of A. In order that X separate points, it is necessary and sufficient that the space E be totally disconnected.*

7. On the set X of semicharacters of the commutative topological semigroup A we define the bicompact-open topology: a basis of neighborhoods is formed by sets of the form

$$U_{C,\delta}(\chi) = \{\chi' \in X;\ |\chi'(a) - \chi(a)| < \delta \text{ for all } a \in C\},$$

where C is a bicompact subset of A.

If the set X is a semigroup with respect to multiplication

$$[\chi_1\chi_2](a) = \chi_1(a)\chi_2(a) \quad (\chi \in X),$$

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then, endowed with the bicompact-open topology, it becomes a commutative topological semigroup. The set of semicharacters of this semigroup will be denoted by A'.

To each element $a \in A$ we associate the element $a' \in A'$: $a'(\chi) = \chi(a)$, $\chi \in X$. The mapping ω, defined by the formula $\omega(a) = a'$, is a homomorphism of A into A'.

In [1] (see also [4]) it is proved that if the semigroup of characters of a bicompact inverse commutative semigroup with identity A separates points, then the duality theorem holds for A. Theorem 3 makes it possible to strengthen this result in the following way.

Theorem 4. *Let A be a bicompact commutative inverse semigroup with identity, whose subsemigroup of idempotents is totally disconnected. Then A and A' are topologically isomorphic under the mapping ω.*

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Note: Figure translations are in progress. See original paper for figures.

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