ON THE ASYMPTOTICS OF SOLUTIONS OF THE DIRICHLET PROBLEM NEAR AN IRREGULAR BOUNDARY

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ON THE ASYMPTOTICS OF SOLUTIONS OF THE DIRICHLET PROBLEM NEAR AN IRREGULAR BOUNDARY

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In this work the asymptotic behavior of solutions of the first boundary-value problem for the equation $-\Delta u = f$ is studied. We restrict ourselves to considering the Laplace operator and the simplest singularities on the boundary of the domain. However, the method admits generalization to second-order elliptic equations with variable coefficients, and also makes it possible to consider cases of more complicated boundary configurations.

1°. Let Ω be a finite domain in R_n with boundary Γ; let the origin $O \in \Gamma$; $\vec{x} = (x_1, \dots, x_{n-1})$; $x = (\vec{x}, x_n)$; $\varphi_x = |x|^{-1}x$; $\cos\theta = |x|^{-1}x_n$; $\Sigma_r = \{x : |x| \le r\}$; σ_r is the boundary of Σ_r; K_r is the projection of $\sigma_r \cap \Omega$ from the point O onto σ_1; $\nu_n = \mathrm{mes}_n\,\Sigma_1$; $s_n = \mathrm{mes}_{n-1}\,\sigma_1$. For definiteness we assume that $\Omega \subset \Sigma_1$, and choose $\delta > 0$ such that $\Omega \cap \delta_\delta \ne \emptyset$.

Everywhere below by a solution we shall mean a solution of the equation $-\Delta u = f$ such that $u \in L_2^1(\Omega \setminus \Sigma_r)$ and $u = 0$ on $\Gamma \setminus \Sigma_r$ for all $r > 0$. We shall call a solution u growing if $u \in L_2^1(\Omega)$. Introduce the notation: $J(r) = \|u(r,\cdot)\|_{L_2(K_r)}$; λ_r, Λ_r are the first and second eigenvalues of the Dirichlet problem for the Beltrami operator in K_r, if $r \le \delta$. For $r > \delta$ put $\lambda_r, \Lambda_r = 0$. Denote by y_1, y_2 the solutions of the equation $y'' + (n-1)y' - r^{-2}\lambda_r y = 0$ such that $y_1 y_2^{-1} \to 0$ as $r \to 0$ and $y_2(1) = 0$.

Theorem 1. a) Let $u \in L_2^1(\Omega)$ be a solution of the equation $-\Delta u = f$, where $f \equiv 0$ in Σ_ε. Then, for $r_1 < r_2 < \varepsilon$, the estimate

$$J(r_1) \le y_1(r_1) y_1^{-1}(r_2)\, J(r_2)$$

is valid. b) Let u be a growing solution of the equation $\Delta u = 0$. Then

$$J(r_1) \ge y_2(r_1) y_2^{-1}(r_2) J(r_2)$$

for $r_1 < r_2 < 1$.

Theorem 2. There exists no more than one growing solution of the equation $\Delta u = 0$ satisfying the condition

$$\lim_{r\to 0} J^{-2}(r) r^{2-n} \int_r^1 \exp\left\{\int_t^1 \frac{\sqrt{4\Lambda_s + (n-2)^2} + \sqrt{4\lambda_s + (n-2)^2}}{2s}\,ds\right\}\frac{dt}{t} = \infty.$$

Such a solution preserves its sign in Ω.

If λ_r "sufficiently regularly" tends to a limit $\lambda_0 \le \infty$ as $r \to 0$, then one can indicate asymptotic formulas for $y_{1,2}$. These formulas, in combination with Theorem 2, make it possible to obtain the following exact conditions for uniqueness of a nontrivial solution of the equation $\Delta u = 0$, satisfying the inequality $J(r) \le c y_2(r)$: a) $(\Lambda_r - \lambda_r) r^{-1} \in L(0,1)$ for $\lambda_0 < \infty$; b) $(\sqrt{\Lambda_r} - \sqrt{\lambda_r}) r^{-1} \in L(0,1)$ for $\lambda_0 = \infty$.

From Theorem 1 one can obtain estimates near the point O for the Green function $G(x,s)$ and the harmonic measure $H(x,E)$, $E \subset \Gamma$, $x \in \Omega$ (cf. ([1])).

Theorem 3. For any $\alpha > 1$, $\beta < 1$ there exists a constant $k = k(\alpha,\beta)$ such that

$$G(x,s) \le k y_1(\alpha|x|) y_1^{-1}(\beta|s|)|s|^{2-n} \quad \text{for } \alpha|x| \le \beta|s|;$$
$$H(x, \Gamma\setminus\Sigma_\rho) \le k y_1(\alpha|x|) y_1^{-1}(\beta\rho) \quad \text{for } \alpha|x| \le \beta\rho.$$

The last inequality makes it possible to obtain an estimate for the modulus of continuity of a harmonic function $v \in C(\overline{\Omega})$ at the point O. Let $\gamma(t)$ be a modulus

continuity of the function $v|_\Gamma$ at the point O. Then, for $x \in \Omega$,

$$|v(x) - v(O)| \le \gamma(\alpha|x|) + k y_1(\alpha|x|) \int_{\alpha|x|}^1 y_1^{-1}(\beta t)\,d\gamma(t).$$

2°. We shall give asymptotic formulas for an increasing solution of the equation $\Delta u = 0$. We confine ourselves here to the case when Ω, in a neighborhood of the point O, is a body of revolution $\{x : \theta \le \theta_r\}$, and θ_r tends to the limit θ_0 as $r \to 0$. Put $\omega_r = \text{mes}_{n-1} K_r$, $\omega_0 = \lim \omega_r$ as $r \to 0$, and $\chi_r = \omega_0 - \omega_r$. In deriving the asymptotic formulas one has to require that the function χ_r and its derivatives satisfy, as $r \to 0$, certain integral decay conditions characterizing the remainder term. Since these general conditions are rather cumbersome, we restrict ourselves here to indicating asymptotics for the particular, but typical, case

$$\chi_r = c r^{c_0} (\ln r)^{c_1} (\ln\ln r)^{c_2} \ldots (\ln_k r)^{c_k *}.$$

Let $\theta_0 \in (0,\pi)$. Then, as $r \to 0$,

$$u = \begin{cases} r^{2-n-m_0}\left[C_{m_r}^{n/2-1}(\cos\theta) + O\left(\int_0^r |\chi_t|\,\frac{dt}{t}\right)\right], & \text{if } \chi_r r^{-1} \in L(0,1) \text{ and } c_0 \le 1; \\ r^{2-n-m_0}\exp\left\{\mu^{-1}\int_r^1 \chi_t\,\frac{dt}{t}\right\}\left[C_{m_r}^{n/2-1}(\cos\theta) + O\left(|\chi_r| + \int_0^r \chi_t^2\,\frac{dt}{t}\right)\right], & \text{if } \chi_r r^{-1} \notin L(0,1),\ \chi_r^2 r^{-1} \in L(0 \end{cases} \tag{1}$$

where $C_m^{n/2-1}$ is the Gegenbauer function, $m = m_r$ satisfies the equation $C_{m_r}^{n/2-1}(\cos\theta_r) = 0$, and

$$\mu = \frac{s_{n-1}(n-2+2m_0)}{(n-2)^2\sin^2\theta_0}\left[C_{m_0-1}^{n/2}(\cos\theta_0)\right]^{-2}\int_{\cos\theta_0}^1 (1-t^2)^{(n-3)/2}\left[C_{m_0}^{n/2-1}(t)\right]^2 dt.$$

If $\theta_0 = 0$ and $\theta_r r^{-1} \in L(0,1)$, then, as $r \to 0$,

$$u = (r\theta_r)^{(2-n)/2}\exp\left\{\nu\int_r^1 \theta_t^{-1}\,\frac{dt}{t}\right\}\left[\left(\frac{\theta_r}{\theta}\right)^{(n-3)/2} J_{(n-3)/2}\left(\nu\frac{\theta}{\theta_r}\right) + O\left(\int_0^r \theta_t\,\frac{dt}{t}\right)\right], \tag{2}$$

where ν is the least positive zero of the Bessel function $J_{(n-3)/2}$.

We pass to the case $\theta_0 = \pi$. If $n = 3$, $r^{-1}\ln^{-1}\chi_r \in L(0,1)$ and $r^{-1}\ln^{-2}\chi_r \in L(0,1)$, then there exists a solution having, as $r \to 0$, the form

$$u = r^{-1}\exp\left\{-\int_r^1 \ln^{-1}\chi_t\,\frac{dt}{t}\right\}\left[1 - \frac{2\ln\cos\theta/2}{\ln\chi_r} + O\left(|\ln^{-1}\chi_r| + \int_0^r \ln^{-2}\chi_t\,\frac{dt}{t}\right)\right]. \tag{3}$$

If $n > 3$ and $\chi_r^{2(n-3)/(n-1)} r^{-1} \in L(0,1)$, then, as $r \to 0$,

$$u = r^{2-n}\exp\left\{\frac{(n-3)s_{n-1}}{(n-2)s_n}\nu_{n-1}^{(3-n)/(n-1)}\int_r^1 \chi_t^{(n-3)/(n-1)}\,\frac{dt}{t}\right\}[1 + o(1)]. \tag{4}$$

Under the same condition, formula (4) admits a refinement, which we do not present because of its cumbersomeness.

If $f(x)$ satisfies the condition $F(r)y_1(r)r^{n-1} \in L(0,1)$, where $F(r) = \sup|f(x)|$ on σ_r, then there exists an increasing solution of the equation $-\Delta u = f$, the principal term of whose asymptotics is determined by formulas (1)–(4).

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We note that for the "spherical" cone $\theta \leq \theta_0 \in (0, \pi)$, asymptotic expansions of solutions of general elliptic problems are given in (2).

For unbounded domains one can obtain analogous asymptotic representations of solutions in a neighborhood of an infinitely distant point. As an example, let us consider a "quasi-cylindrical" domain Ω. Let Ω_t be the section of Ω by the hyperplane $x_n = t \geq 0$. We shall assume that the projection of Ω_t onto $x_n = 0$ is obtained from Ω_0 by a similarity transformation with respect to O with coefficient $\alpha(t)$ and shift by the vector $\vec{\beta}(t) = (\beta^{(1)}(t), \dots, \beta^{(n-1)}(t))$. Because of the cumbersomeness of the exact conditions and of the remainder term, in the general case we give the formula only for the case when α and $\beta^{(i)}$ are functions of the form $ct^{c_0}(\ln t)^{c_1} \cdots (\ln_k t)^{c_k}$. Namely, if
$\alpha a'^2(t)\alpha^{-1}(t) \in L(0, \infty)$, $|\vec{\beta}|^2\alpha^{-2} < c < \infty$, then there exists a solution of the equation $\Delta u = 0$, $u = 0$ on Γ, such that as $x \to \infty$

$$u(x) = \alpha^{(2-n)/2}(x_n) \exp\left\{ \lambda_0^{1/2} \int_0^{x_n} \frac{dt}{\alpha(t)} \left[\Psi\left(\frac{\vec{\mathcal{X}} - \vec{\beta}(x_n)}{\alpha(x_n)} \right) + O\left(|\alpha'(x_n)| + \int_{x_n}^{\infty} \frac{\alpha a'^2}{\alpha}\, dt \right) \right] \right\},$$

where λ_0 is the first eigenvalue, and $\Psi(\vec{\mathcal{X}})$ is the eigenfunction of the Dirichlet problem for the Laplace operator in Ω_0. This formula generalizes, to the case $n > 2$, the asymptotic formula for a conformal mapping of an infinite strip, obtained in ([3]) under less stringent conditions.

Let us return to the case of a bounded domain Ω. It can be shown that increasing solutions of the equation $\Delta u = 0$ admitting asymptotic representations of the type (1)–(4) have the form

$$u(x) = \lim_{s \to 0} G(x, s) G^{-1}(x_0, s),$$

where $x_0 \in \Omega$, and $s \to 0$ along a nontangential path*. Hence it follows that

$$\lim_{s \to 0} \lim_{x \to 0} G^{-1}(x, s) y_1(|x|) \Psi_{|x|}(\varphi_x) y_2(|s|) \Psi_{|s|}(\varphi_s) = c \in (0, \infty),$$

where $\Psi_{|y|}(\varphi_y)$ is the first eigenfunction of the Dirichlet problem for the Beltrami operator in $K_{|y|}$, and $x, s \to 0$ along nontangential paths.

3°. Let us note some applications of the results obtained. First consider the question of the deficiency index of the operator Δ. The case $n = 2$ was studied in ([4,5]). Here and below we assume that the boundary Γ is sufficiently regular everywhere except at the point O. Let the operator Δ be defined on
$\mathcal{D}(\Delta) = L_2^2(\Omega) \cap \dot{L}_2^1(\Omega)$, and let $\overline{\Delta}$ be the closure of Δ, $\widetilde{\Delta}$ the Friedrichs extension

of Δ, and Δ^* the operator adjoint to Δ. From Theorem 1 it follows that $\overline{\Delta} = \widetilde{\Delta}$ if $n > 3$, or if $n = 3$ and $ry_2(r) \in L_2(0, 1)$. Under certain qualitative restrictions on Γ the latter condition is also necessary. If it is violated, the deficiency index is equal to 1. In this case the asymptotic behavior of the solution of the equation $\Delta^* u = 0$ is described by formulas of the type (1)–(4). If $\overline{\Delta} \neq \widetilde{\Delta}$, then $v \in \mathcal{D}(\Delta^*)$ belongs to $\mathcal{D}(\widetilde{\Delta})$ if and only if $v(s)G^{-1}(x_0, s) \to c < \infty$ as $s \to 0$ along a nontangential path. Moreover, v belongs to $\mathcal{D}(\overline{\Delta})$ if and only if $c = 0$. If $n = 3$ and Γ near the point O is a cone with regular boundary, then $\Delta = \overline{\Delta} = \widetilde{\Delta}$ for $\lambda_0 > 3/4$, $\Delta = \overline{\Delta} \neq \widetilde{\Delta}$ for $\lambda_0 < 3/4$, and $\Delta = \overline{\Delta}$, $\dim \mathcal{D}(\overline{\Delta}) \,(\mathrm{mod}\, \mathcal{D}(\Delta)) = \infty$ for $\lambda_0 = 3/4$.

Let Ω in a neighborhood of the point O be a body of revolution, and suppose $(\frac{1}{2}s_n - \omega_r)^2 r^{-1} \in L(0, 1)$. Then the condition

$$\varliminf_{r\to 0} \int_r^1 (\tfrac{1}{2}s_n - \omega_t)\frac{dt}{t} < +\infty$$

is necessary and sufficient in order that all harmonic functions,

* This means that $\varliminf \rho d^{-1} > 0$, where ρ is the distance from s to $\Gamma \cap \sigma_{|s|}$, d is the diameter of $\Omega \cap \sigma_{|s|}$.

having a minimum at the point O, satisfy the inequality $\lim |x|^{-1}(u(x) - u(0)) > 0$ as $x \to 0$ along a non-tangential path (cf. (6)).

In conclusion let us consider the question of the validity of the estimate $\|D^2 u\|_p \leq c\|\Delta u\|_p$ (cf. (7)) under the assumption that Γ near O is a cone with regular boundary. This estimate holds for all $p > 1$ if $\lambda_0 \geq 2n$. If, however, $\lambda_0 < 2n$, then it is valid for

$$p < 2n\left(n + 2 - \sqrt{4\lambda_0 + (n-2)^2}\right)^{-1};$$

the only exception is the case of a regular boundary when $\lambda_0 = n - 1$, but the estimate is valid for all $p > 1$.

Let us also note that the results obtained in 1°, 2° turn out to be useful in proving pointwise estimates for derivatives of solutions, as well as assertions of the type of the Phragmén–Lindelöf principle.

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Note: Figure translations are in progress. See original paper for figures.