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**Abstract**

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*MATHEMATICS*

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**ON AN INVARIANT EQUIPPING OF A SURFACE OF A SPACE WITH AFFINE CONNECTION**

*(Presented by Academician A. D. Aleksandrov on 4 VI 1966)*

An invariant equipping of a surface of a space with affine connection (with curvature and torsion) was considered earlier only for hypersurfaces  $(^{1-3})$ , for two-dimensional surfaces  $(^4)$ , and also for surfaces  $X_m$  of spaces  $L_n$ , whose dimension  $n$  is equal to the dimension of the osculating space to  $X_m$  of some order  $p = 2, 3, 4, \dots$   $(^5)$ ; G. F. Laptev solved this problem for  $n < m(m + 3)/2$ . In the present work a method is indicated for internally equipping a surface  $X_m$  of spaces  $L_n$  of arbitrary dimension.

Let  $X_m$  be given in  $L_n$ , with connection object  $\Gamma_{jk}^i(x^l)$  ( $i, j, k, l, \dots = 1, 2, \dots, n$ ), by the equations

$$x^i = x^i(u^1, u^2, \dots, u^m), \tag{1}$$

where the functions (1) and  $\Gamma_{jk}^i$  are assumed differentiable a sufficient number of times. As in  $(^3, ^4)$ , put

$$B_\alpha^i \equiv \partial_\alpha x^i, \quad B_{\alpha\beta}^i \equiv \partial_\beta \partial_\alpha x^i \quad \text{and so on;}$$

$$\bar{B}_{\alpha\beta}^i \equiv B_{\alpha\beta}^i + \Gamma_{(jk)}^i B_\alpha^j B_\beta^k, \quad \bar{B}_{\alpha\beta\sigma}^i \equiv \partial_\sigma \bar{B}_{\alpha\beta}^i + \Gamma_{(jk)}^i \bar{B}_{\alpha\beta}^j B_\sigma^k \quad \text{and so on}$$

$$(\alpha, \beta, \sigma, \dots = 1, 2, \dots, m).$$

$B_\alpha^i$  is a tensor both in  $X_m$  and in  $L_n$ ;  $\bar{B}_{\alpha\beta}^i$  is a tensor only in  $L_n$ . Let  $T^{(1)}\{B_\alpha^i\}$ ,  $T^{(2)}\{\bar{B}_{\alpha\beta}^i, B_\alpha^i\}, \dots, T^{(p-1)}\{\bar{B}_{\alpha_1 \dots \alpha_{p-1}}^i, \dots, B_\alpha^i\}$  be the spaces tangent to  $X_m$  of orders  $1, 2, \dots, p - 1$ , where  $p$  is such a natural number that  $\dim T^{(p-1)} \leq n$ , while  $\dim T^{(p-1)}$  plus the number of vectors  $\bar{B}_{\alpha_1 \dots \alpha_p}^i$  is greater than  $n$ . We shall

assume that each  $T^{(q)}$  ( $q \leq p-1$ ) has maximal dimension, i.e. that the vectors (relative to  $L_n$ )  $B_\alpha^i, \dots, \bar{B}_{\alpha_1, \dots, \alpha_{p-1}}^i$  are linearly independent.

Consider the system of linear equations in  $m_i$

$$B_\alpha^i m_i = 0, \dots, \bar{B}_{\alpha_1, \dots, \alpha_{p-1}}^i m_i = 0. \quad (2)$$

It is invariant with respect to coordinate transformations both in  $X_m$  and in  $L_n$  and has  $N = n - \dim T^{(p-1)}$  linearly independent solutions  $m_i^a$  ( $a, b, \dots = 1, 2, \dots, N$ ). Then

$$\omega_{\alpha_1 \dots \alpha_p}^a \equiv \bar{B}_{\alpha_1 \dots \alpha_p}^i m_i^a \quad (3)$$

forms a system of  $N$  linearly independent tensors in  $X_m$ .

If one chooses another system of solutions of (2),

$${}^* m_i^b = \lambda_a^b m_i^a \quad (\det |\lambda_a^b| \neq 0),$$

then  $\omega_{\alpha_1 \dots \alpha_p}^a$  also undergoes the linear transformation

$${}^* \omega_{\alpha_1 \dots \alpha_p}^b = {}^b \lambda \omega_{\alpha_1 \dots \alpha_p}^a, \quad (4)$$

i.e., the index  $a$  for  $\omega_{\alpha_1 \dots \alpha_p}^a$  has, with respect to the transformations (4), a tensorial character.

Put

$$A^{a_1 \dots a_m} \equiv \omega_{\underline{a_1} \dots \underline{a_p}}^{a_1} \dots \omega_{\underline{a_1} \dots \underline{a_p}}^{a_m}, \quad (5)$$

where alternation is performed over the indices underlined by identical signs. Obviously,  $A^{a_1 \dots a_m}$  ( $a_\alpha = 1, 2, \dots, N$ ) is a tensor with respect to the transformations (4) (a  $\lambda$ -tensor) and a scalar density of weight  $p$  in  $X_m$ . We shall now distinguish two cases.

I. If  $p$  is even, then  $A^{a_1 \dots a_m}$  is symmetric. We shall assume that it has nonzero discriminant. Then, as follows from the works (<sup>7-10</sup>), there exists a  $\lambda$ -tensor  $\tilde{A}_{a_1 \dots a_m}$  such that

$$\tilde{A}_{aa_2 \dots a_m} A^{ba_2 \dots a_m} = \delta_a^b.$$

In  $X_m$  it is a tensor density of weight  $-p$ .<sup>\*</sup> We now define in  $X_m$  a symmetric tensor density of weight  $-p$ :

$$\mathfrak{H}_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m} \equiv \omega_{(\underline{a'_1} \dots \underline{a'_p})}^{a_1} \dots \omega_{(\underline{a'_1} \dots \underline{a'_p})}^{a_m} \tilde{A}_{a_1 \dots a_m}. \quad (6)$$

Assuming the discriminant of this density to be nonzero and dividing it by a suitable power of its discriminant, we obtain a “pure” tensor

$$H_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m}.$$

In this case, by the method of Lieber–Ermakov–Kramlet (see (7–9)), we can invariantly associate with  $H_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m}$  a connection  $\tilde{\Gamma}_{\alpha\beta}^\sigma$ , which is invariant in  $L_n$ .

II. If  $p$  is odd, then  $A^{a_1 \dots a_m}$  is skew-symmetric. Consider the system of linear homogeneous equations with respect to  $C^{\alpha_1 \dots \alpha_p \alpha_{p+1}}$ :

$$C^{\alpha_1 \dots \alpha_p \alpha_{p+1}} \omega_{\alpha_1 \dots \alpha_p}^a = 0 \quad (\alpha_1 \dots \alpha_{p+1} = 1, \dots, m). \quad (7)$$

It is easy to calculate that the number  $k$  of equations in this system and the number  $l$  of unknowns are connected by the condition  $k \leq l$ . Indeed,  $l$  is equal to the number of combinations with repetitions from  $m$  taken  $p$  at a time, multiplied by  $m$ :  $l = \tilde{C}_m^p m$ ;  $k = Nm = (n - \dim T^{(p-1)})m \leq$  (the number of vectors  $B_{\alpha_1 \dots \alpha_p}^i + \dim T^{(p-1)} - \dim T^{(p-1)})m$ , or  $k \leq \tilde{C}_m^p m$ , i.e.  $k \leq l$ . Equality is attained only in the case  $n = \dim T^{(p)}$ , for which the connection in  $X_m$  is introduced by the method of Hlavatý (see (5)). Excluding this case, we have  $k < l$ , and the system (7) has a certain number  $K$  of linearly independent solutions

$$C_u^{\alpha_1 \dots \alpha_p \alpha_{p+1}} \quad (v, u = 1, 2, \dots, K).$$

These tensors\*\* are determined up to a linear transformation:

$$*C_u^{\alpha_1 \dots \alpha_p \alpha_{p+1}} = v \Lambda C_v^{\alpha_1 \dots \alpha_p \alpha_{p+1}}. \quad (8)$$

\* To form a pseudotensor “reciprocal” to a given pseudotensor, one may use any of its relative invariants; for example, its multidimensional determinant is convenient (see (10)).

\*\* The tensorial character of the quantities  $C_u^\alpha$  is established directly.

Again we shall say that  $C_u^a \dots$  relative to  $u$  is a  $\Lambda$ -tensor. Analogously to (5), introduce the  $\Lambda$ -tensor

$$B_{u_1 \dots u_m} \equiv C_{u_1}^{\hat{\alpha}_1 \dots \hat{\alpha}_{p+1}} \dots C_{u_m}^{\hat{\beta}_1 \dots \hat{\beta}_{p+1}}, \quad (9)$$

where the hats above again indicate alternation. In this case  $B_{u_1 \dots u_m}$  will be symmetric. Assuming the discriminant of (9) to be nonzero, we construct for it the reciprocal  $\Lambda$ -tensor  $\tilde{B}^{u_1 \dots u_m}$ , and then the tensor density in  $\dot{X}_m$

$$\mathfrak{h}^{\alpha'_1 \dots \alpha'_p \dots \alpha_1^m \dots \alpha_p^m} \equiv C_{u_1}^{(\alpha'_1 \dots \alpha'_p)} \dots C_{u_m}^{\alpha_1^m \dots \alpha_p^m} \tilde{B}^{u_1 \dots u_m}, \quad (10)$$

which is again sufficient for introducing a connection  $\tilde{\Gamma}_{\alpha\beta}^\sigma$  in  $\dot{X}_m$ .

We note that the differentiability classes of the functions  $\Gamma_{jk}^i$  and (1), used for constructing  $\tilde{\Gamma}_{\alpha\beta}^\sigma$ , are respectively equal to  $C^{p-1}$  and  $C^{p+1}$ . For  $X_m$  in  $L_n$ , where  $n < m(m+3)/2$ , we obtain  $C^1$  and  $C^3$ .

Introduce the following system of quantities:

$$\tilde{\omega}_a^{\sigma_1 \dots \sigma_p} \equiv \frac{1}{m!} \omega_{a'_1 \dots a'_p}^{a_1} \dots \omega_{\alpha_1^{m-1} \dots \alpha_p^{m-1}} A_{\alpha_1 \dots \alpha_{m-1} a} \delta_{1 \dots m}^{\alpha'_1 \dots \alpha_1^{m-1} \sigma_1} \dots \delta_{1 \dots m}^{\alpha'_p \dots \alpha_p^{m-1} \sigma_p},$$

where  $\delta_{1 \dots m}^{\alpha'_1 \dots \alpha_1^{m-1} \sigma_1}$  (the indices range from 1 to  $m$ ) is the generalized Kronecker symbol (contraction of a tensor with  $\frac{1}{m!} \delta_{1 \dots m}^{\alpha_1 \dots \alpha_m}$  is equivalent to its alternation with respect to the indices  $\alpha_1 \dots \alpha_m$ ). It is easy to see that

$$\tilde{\omega}_a^{\sigma_1 \dots \sigma_p} \omega_{\sigma_1 \dots \sigma_p}^b = \delta_a^b \quad (a, b = 1, \dots, N). \quad (11)$$

We define the invariant equipment by the spaces

$$\mathfrak{P}^{(2)}\{D_\beta B_a^i\}, \quad \mathfrak{P}^{(3)}\{D_\beta D_\gamma B_\delta^i\}, \dots, \quad \mathfrak{P}^{(p-1)}\{D_{\sigma_1} \dots D_{\sigma_{p-2}} B_{\sigma_{p-1}}^i\}$$

(spanned by the vectors indicated in braces) and by the system of vectors

$$n_a^i \equiv \tilde{\omega}^{\sigma_1 \dots \sigma_p} D_{\sigma_1} \dots D_{\sigma_{p-1}} B_{\sigma_p}^i.$$

Here  $D_\sigma$  is the operator of mixed differentiation with respect to the pair of connections  $\Gamma_{jk}^i$ ,  $\tilde{\Gamma}_{\alpha\beta}^\sigma$ , where  $\tilde{\Gamma}_{\alpha\beta}^\sigma$  has been introduced by the method described above. It is easy to see that, by virtue of the assumptions made at the beginning of the paper and of (11), we obtain a genuinely invariant equipment. The connection  $\Gamma_{\alpha\beta}^\sigma$  induced by this equipment, generally speaking, does not coincide with  $\tilde{\Gamma}_{\alpha\beta}^\sigma$ .

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## CITED LITERATURE

1. V. Hlavaty, Math. Zs., 38, 283 (1934).
2. F. Nožička, Casopis pro past. mat. a fys., 75, No. 3, 179 (1950).
3. V. D. Izmailov, UMN, 15, issue 5 (95), 171 (1960).
4. V. D. Izmailov, Matem. sborn., 54 (95), No. 3, 312 (1961).
5. V. Hlavaty, Akad. wetensch. Amsterdam, Proc. Section Sci., 52, No. 5-8 (1949).
6. G. F. Laptev, DAN, 126, No. 3, 490 (1959).
7. T. Kramlet, Ann. Math., 31, 134 (1930).
8. A. E. Liber, Tr. seminara po vektorn. i tenzorn. analizu, 9, 319 (1952).
9. Yu. I. Ermakov, DAN, 128, No. 3, 460 (1959).
10. K. Weise, Math. Zs., 43, 44 (1937).

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