WEIGHT SPACES OF FUNCTIONS WITH DOMINANT MIXED DERIVATIVES AND DIFFERENTIAL PROPERTIES OF FUNCTIONS FROM THESE SPACES

1967

SovietRxiv

Abstract

Full Text

UDC 513.881+517.51

MATHEMATICS

A. D. DZHABRAILOV

WEIGHT SPACES OF FUNCTIONS WITH DOMINANT MIXED DERIVATIVES AND DIFFERENTIAL PROPERTIES OF FUNCTIONS FROM THESE SPACES

(Presented by Academician S. L. Sobolev on 19 IX 1966)

The investigation, by the variational method, of hyperelliptic equations degenerating on the boundary has led to the necessity of studying the boundary properties of functions from weight spaces of functions with dominant mixed derivatives, which is what is done in the present work. These spaces in the unweighted case were studied in the papers ([3,5,9,10]), and the general theory of weight spaces is set forth in ([3]).

Let E^n be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. $\overset{+}{E}{}^n = \{x;\ x_i > 0\ (i = 1, \dots, n)\}$. Suppose that e is an arbitrary subset of the set of natural numbers $e_n = \{1, \dots, n\}$. If $K = (k_1, \dots, k_n)$ is a given vector, then, for each fixed subset e, let $K^e = (k_1^e, \dots, k_n^e)$, where $k_j^e = k_j$ for $j \in e$ and $k_j = 0$ for $j \in e_n \setminus e$. The carrier of the vector K is called the smallest subset e of the set e_n such that $K^e = K$, and it is denoted by e_k.

Let $r = (r_1, \dots, r_n)$ be a vector with nonnegative components and with nonempty carrier e_r. Put $r_i = \bar{r}_i + \beta_i$, where $\bar{r}_i$ is the integral part of r_i, so that $0 \le \beta_i < 1$, and also $r_i = \bar{\bar{r}}_i + \gamma_i$, where $\bar{\bar{r}}_i$ is the greatest integer less than r_i, consequently $0 < \gamma_i \le 1$. If $r_i = 0$, then put $\bar{r}_i = 0$ and $\bar{\bar{r}}_i = 0$. Thus, to each vector $r = (r_1, \dots, r_n)$ there correspond the vectors $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$ and $\bar{\bar{r}} = (\bar{\bar{r}}_1, \dots, \bar{\bar{r}}_n)$.

Let $f(x)$ be a sufficiently smooth function defined in E^n. By $\Delta_j^{k_j}(t_j) f(x)$ we denote the finite difference of the function f of order k_j with respect to the variable x_j with step t_j. For each subset e of e_n and for integer-valued vectors $k = (k_1, \dots, k_n)$ and $m = (m_1, \dots, m_n)$, put

$$\Delta^{K^e}(t) f(x) = \left[\prod_{j\in e} \Delta_j^{k_j}(t_j)\right] f(x),$$

$$D^{m^e} f(x) \equiv D_1^{m_1^e} \cdots D_n^{m_n^e} f(x) = \frac{\partial^{m_1^e}}{\partial x_1^{m_1^e}} \cdots \frac{\partial^{m_n^e}}{\partial x_n^{m_n^e}} f(x).$$

Let e_r^* be the set of indices from e_r such that, for $i \in e_r^*$, r_i is not an integer; let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a vector whose components satisfy the conditions $\alpha_i > -1$ $(i = 1, \dots, n)$.

For any fixed e from e_r, put, when $e^* \equiv e \cap e_r^* = \emptyset$,

$$\|f, L_{p,\alpha}^{r^e}(\overset{+}{E}{}^n)\| = \|D^{r^e} f, L_{p,\alpha}(\overset{+}{E}{}^n)\|,$$

and when $e^* \equiv e \cap e_r^* \neq \emptyset$,

$$\|f, L_{p,\alpha}^{r^e}(\overset{+}{E}{}^n)\| = \|D^{\bar{\bar{r}}^e} f, L_{p,\alpha}(\overset{+}{E}{}^n)\| + I_{e,\alpha}(f),$$

where

$$\|f, L_{p,\alpha}(\overset{+}{E}{}^n)\| = \left(\int_{\overset{+}{E}{}^n} \prod_{j=1}^{n} x_j^{\alpha_j} |f(x)|^p \, dx \right)^{1/p};$$

$$I_{e,\alpha}(f) = \left(\int_0^\infty \cdots \int_0^\infty \left\| \Delta^{\omega e^*}(t) D_r^{\bar{e}} f,\ L_{p,\alpha}(\overline{E}^n) \right\|^p \prod_{j \in e^*} \frac{dt_j}{t_j^{1+\beta_{jp}}} \right)^{1/p};$$

$\omega = (1, \dots, 1)$ is a vector all of whose components are equal to one; $1 \leq p \leq \infty$.

Definition 1. The space $S_{p,\alpha}^r W(\overline{E}^n)$ will mean the closure of the set of smooth finite functions in E^n with respect to the norm

$$\left\| f,\ S_{p,\alpha}^r W(\overline{E}^n) \right\| = \sum_{e \subseteq e_r} \left\| f,\ L_{p,\alpha}^{re}(\overline{E}^n) \right\|,$$

where the sum is taken over all possible subsets e of the set e_r. Further, for each e from e_r let

$$\left\| \dot{f},\ \mathcal{L}_{p,\alpha}^{re}(\overline{E}^n) \right\| = \left(\int_0^\infty \cdots \int_0^\infty \left\| \Delta^{2\omega e}(t) D_r^{\bar{e}} f,\ L_{p,\alpha}(\overline{E}^n) \right\|^p \prod_{j \in e} \frac{dt_j}{t_j^{1+\gamma_{jp}}} \right)^{1/p}.$$

Definition 2. The space $S_{p,\alpha}^r B(\overline{E}^n)$ will mean the closure of the set of smooth finite functions in E^n with respect to the norm

$$\left\| f,\, S^r_{p,\alpha} B(\overline{E}^n) \right\| = \sum_{e \subseteq e_r} \left\| f,\, \mathcal{L}^{re}_{p,\alpha}(\overline{E}^n) \right\|.$$

Definition 3. The space $S^r_{p,\alpha}\mathcal{H}(\overline{E}^n)$ will mean the closure of the set of smooth finite functions in E^n with respect to the norm

$$\left\| f,\, S^r_{p,\alpha}\mathcal{H}(\overline{E}^n) \right\| = \sum_{e \subseteq e_r} M^e_\alpha(f),$$

where

$$M^e_\alpha(f) = \sup_{t^e} \prod_{j \in e} t_j^{-\gamma_j} \left\| \Delta^{2\omega e}(t) D^{\bar{e}}_r f,\, L_{p,\alpha}(\overline{E}^n) \right\|.$$

Theorem 1. Let: 1) $1 < p \le q < \infty$; 2) $r = (r_1, \dots, r_n)$ be a vector with nonnegative components and with nonempty support e_r; if $q > p$, then $e_r = e_n$; 3) $\nu = (\nu_1, \dots, \nu_n)$ be a vector with integer nonnegative components with support $e_\nu \subseteq e_r$; 4) $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be two vectors with supports $e_\mu \subseteq e_\alpha \subseteq e_r$, and the components of these vectors are related as follows: $\alpha_j/p \ge \mu_j/q \ge 0$ for $j \in e_\mu$ and $\alpha_j \ge 0$ for $j \in e_\alpha \setminus e_\mu$; 5) $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be a vector with support $e_\varepsilon = e_r$, where

$$\varepsilon_j = r_j - \nu_j - ((1+\alpha_j)/p - (1+\mu_j)/q) > 0 \quad (j \in e_r);$$

6) $f \in S^r_{p,\alpha} W(\overline{E}^n)$.

Then $D^\nu f \in S^\rho_{q,\mu} W(\overline{E}^n)$, where $\rho = (\rho_1, \dots, \rho_n)$ is a vector with support $e_\rho \subseteq e_r$ and with components $0 < \rho_j \le \varepsilon^*_j$ for $j \in e^*_r \cap e_\rho$ and $0 < \rho_j < \varepsilon_j$ for $j \in e_\rho \setminus (e^*_r \cap e_\rho)$, and the inequality holds

$$\left\| D^\nu f,\, S^\rho_{q,\mu} W(\overline{E}^n) \right\| \le C \left\| f,\, S^r_{p,\alpha} W(\overline{E}^n) \right\|,$$

where C is a constant independent of f.

We shall denote the results of this theorem by the relations

$$S^r_{p,\alpha} W(\overline{E}^n) \to S^\rho_{q,\mu} W(\overline{E}^n).$$

Theorem 2. Suppose the conditions 1)–5) of Theorem 1 are satisfied. Then the embedding holds

$$S^r_{p,\alpha} B(\overline{E}^n) \to S^{\rho^*}_{q,\mu} B(\overline{E}^n),$$

where $\rho^* = (\rho_1^*, \dots, \rho_n^*)$ is a vector with support $e_{\rho^*} \subseteq e_r$ and $0 < \rho_j^* \le \varepsilon_j$ $(j \in e_{\rho^*})$.

Theorem 3. Under conditions 1)–5) of Theorem 1, the embedding

$$S^r_{p,\alpha}\mathcal{H}(\overset{+}{E}{}^n) \to S^{\rho^*}_{q,\mu}\mathcal{H}(\overset{+}{E}{}^n),$$

holds, where $\rho^* = (\rho_1^*, \dots, \rho_n^*)$ is the same vector as in Theorem 2.

Theorem 4. Under conditions 1)–5) of Theorem 1, the embedding

$$S^r_{p,\alpha}\mathcal{W}(\overset{+}{E}{}^n) \to S^{\rho^{**}}_{q,\mu}B(\overset{+}{E}{}^n),$$

holds, where $\rho^{**} = (\rho_1^{**}, \dots, \rho_n^{**})$ is a vector with support $e_{\rho^{**}} \subseteq e_r$, and $0 < \rho_j^{**} \le \varepsilon_j$ for $j \in e_{\rho^{**}} \cap e_r^*$, and $0 < \rho_j^{**} < \varepsilon_j$ for $j \in e_{\rho^{**}} \setminus (e_{\rho^{**}} \cap e_r^*)$.

Theorem 5. Under conditions 1)–5) of Theorem 1, the embedding

$$S^r_{p,\alpha}B(\overset{+}{E}{}^n) \to S^{\rho^*}_{q,\mu}\mathcal{H}(\overset{+}{E}{}^n),$$

holds, where the vector ρ^* is the same as in Theorems 2 and 3.

Below we give several theorems characterizing the boundary properties of functions from the corresponding weighted functional spaces.

Theorem 6. Let: 1) $1 < p \le q < \infty$; 2) $r = (r_1, \dots, r_n)$ be a vector with positive components; 3) m be a natural number $\le n$; 4) $\nu = (\nu_1, \dots, \nu_n)$ be a vector with nonnegative integer components; 5) $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\mu = (\mu_1, \dots, \mu_m)$ be, respectively, n- and m-dimensional vectors, where $\alpha_j/p \ge \mu_j/q \ge 0$ $(j = 1, \dots, m)$ and $\alpha_i > -1$ $(i = m+1, \dots, n)$; 6) $\chi = (\chi_1, \dots, \chi_n)$ be an n-dimensional vector with components

$$\chi_j = r_j - \nu_j - ((1+\alpha_j)/p - (1+\mu_j)/q) > 0 \quad (j = 1, \dots, m),$$

$$\chi_i = r_i - \nu_i - (1+\alpha_i)/p > 0 \quad (i = m+1, \dots, n);$$

7) $f \in S^r_{p,\alpha}\mathcal{W}(\overset{+}{E}{}^n)$.

Then, for $x_{m+1} = 0, \dots, x_n = 0$, with respect to the variables $x_1, \dots, x_m$, the function

$$D^\nu f \in S^\rho_{q,\mu}\mathcal{W}(\overset{+}{E}{}^m), \qquad \rho = (\rho_1, \dots, \rho_m),$$

where $0 < \rho_j \le \chi_j$ for $j \in e_\rho^* \cap e_r^*$, and $0 < \rho_j < \chi_j$ for $j \in e_\rho \setminus (e_\rho^* \cap e_r^*)$; moreover, the inequality

$$\|D^\nu f, S^\rho_{q,\mu}\mathcal{W}(\overset{+}{E}{}^m)\| \le C\|f, S^r_{p,\alpha}\mathcal{W}(\overset{+}{E}{}^n)\|$$

holds, where C is a constant independent of f.

For $m = n$, this theorem is a special case of Theorem 1.

Theorem 7. Suppose that conditions 1)–6) of Theorem 6 hold; in addition, let $f \in S^r_{p,\alpha}B(\overset{+}{E}{}^n)$.

Then, for $x_{m+1} = 0, \dots, x_n = 0$, with respect to the variables $x_1, \dots, x_m$, the function

$$D^\nu f \in S^{\rho^*}_{q,\mu}B(\overset{+}{E}{}^m), \qquad \rho^* = (\rho_1^*, \dots, \rho_m^*),$$

where $0 < \rho_j^* \le \chi_j$ $(j = 1, \dots, m)$; moreover, the inequality

$$\|D^\nu f, S^{\rho^*}_{q,\mu}B(\overset{+}{E}{}^m)\| \le C\|f, S^r_{p,\alpha}B(\overset{+}{E}{}^n)\|$$

holds, where C is a constant independent of f.

For $m = n$, this theorem can be obtained as a special case of Theorem 2.

Theorem 8. Suppose that conditions 1)–6) of Theorem 6 are satisfied. In addition, let

$$f \in S^r_{p,\alpha}\mathcal{H}(\overset{+}{E}{}^n).$$

Then, for $x_{m+1} = 0, \dots, x_n = 0$, as a function of the variables $x_1, \dots, x_m$,

$$D^\nu f \in S^{\rho^*}_{q,\mu}\mathcal{H}(\vec{E}^m),$$

where ρ^* is the same vector as in Theorem 7, and the inequality

$$\|D^\nu f,\, S^{\rho^*}_{q,\mu}\mathcal{H}(\vec{E}^m)\| \le C\|f,\, S^r_{p,\alpha}\mathcal{H}(\vec{E}^n)\|,$$

holds, where C is a constant independent of f.

For $m = n$ this theorem is a special case of Theorem 3.

Finally, using Theorem 4 of S. M. Nikol' skii [(3)] and Theorem 3 of this note, one can assert:

Theorem 9. Let f belong simultaneously to all the spaces

$$S^{r^i}_{p\alpha^i}\mathcal{H}(\vec{E}^{\,n}) \quad (i = 1, \dots, N),$$

where $r^i = (r^i_1, \dots, r^i_n)$ $(i = 1, \dots, N)$ are vectors with nonnegative components and, respectively, with nonempty supports e_{r^i} $(i = 1, \dots, N)$; $\alpha^i = (\alpha^i_1, \dots, \alpha^i_n)$ $(i = 1, \dots, N)$ are vectors with nonnegative components and, respectively, with supports $e_{\alpha^i} \subseteq e_{r^i}$ $(i = 1, \dots, N)$; $1 < p < \infty$. Let $\lambda_i \geq 0$ $(i = 1, \dots, N)$,

$$\sum_{i=1}^{N} \lambda_i \leq 1.$$

Then

$$f \in S_p^{r - \frac{1}{p}\alpha} H(\vec{E}^{\,n}) \quad (\text{see } (3)),$$

where

$$r = \sum_{1}^{N} \lambda_i r^i, \qquad \alpha_i = \sum_{1}^{N} \lambda_i \alpha^i,$$

and the inequalities

$$\|f,\, S_p^{r-\frac{1}{p}\alpha} H\| \leq C \sum_{i=1}^{N} \|f,\, S^{r^i}_{p,\alpha^i}\mathcal{H}\|, \qquad \|f,\, S_p^{r-\frac{1}{p}\alpha} H\| \leq \bar{C} \prod_{i=1}^{N} \|f,\, S^{r^i}_{p,\alpha^i}\mathcal{H}\|^{\lambda_i},$$

hold, where C and $\bar{C}$ are constants independent of f.

The spaces $S^r_{p,\alpha}\mathcal{W}$ with $e_\alpha = \emptyset$, i.e. in the unweighted case, coincide with the spaces S^{rW}_p, first defined, for $e^*_r = e_r$, by S. M. Nikol'skii (5), and for $e^*_r \subseteq e_r \subseteq e_n$, by the author (8,9). In the unweighted case the spaces $S^r_{p,\alpha}B$ coincide with the known spaces S^{rB}_p (see (7–9)), and the spaces $S^r_{p,\alpha}\mathcal{H}$ with spaces of type S^{rH}_p of S. M. Nikol' skii (3). Theorem 6 for $e^*_r = e_r = e_n$ was proved by the author in (10). Theorem 9 in the unweighted case belongs to S. M. Nikol' skii (3). The proofs of the results presented above, based on the new integral representation obtained in the author' s work (10), are carried out by the method of integral representations (see (1,6,10)).

In solving the indicated problems the author used works (1–11).

I express my deep gratitude to Prof. L. D. Kudryavtsev for valuable advice and constant attention to the work.

Moscow Institute of Physics and Technology

Received
23 VIII 1966

REFERENCES

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.
[2] L. D. Kudryavtsev, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, **55** (1959).
[3] S. M. Nikol' skii, Siberian Math. Journal, **4**, No. 6 (1963).
[4] L. D. Kudryavtsev, DAN, **153**, No. 3 (1963).
[5] S. M. Nikol' skii, Mat. Sbornik, **61** (103), 2 (1963).
[6] V. P. Il' in, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, **66** (1962).
[7] A. D. Dzhabrailov, DAN, **129**, No. 2 (1964).
[8] A. D. Dzhabrailov, DAN, **166**, No. 6 (1966).
[9] A. D. Dzhabrailov, Reports of the Academy of Sciences of the Azerbaijan SSR, No. 4 (1966).
[10] A. D. Dzhabrailov, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, **89**, 70–79 (1967).
[11] A. D. Dzhabrailov, ibid., pp. 80–118.

Note: Figure translations are in progress. See original paper for figures.