ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF CERTAIN HYPERGEOMETRIC \(E\)-FUNCTIONS

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Abstract

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ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF CERTAIN HYPERGEOMETRIC E-FUNCTIONS

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In the works of A. B. Shidlovskii ([1,2]) a number of general theorems were proved on the transcendence and algebraic independence of values at algebraic points of E-functions*, which are solutions of linear differential equations with polynomial coefficients. These theorems reduce the proof of algebraic independence of values of E-functions to the proof of algebraic independence of the corresponding functions over the field of rational functions.

One of the methods for proving algebraic independence of functions is a method based on the arithmetic properties of the coefficients of their power series. This method was used by K. Siegel ([4]) and A. B. Shidlovskii ([3]) to prove the transcendence of values of a number of concrete E-functions. In particular, A. B. Shidlovskii in ([3]) applied a generalization of this method to prove the algebraic independence of the values of the functions

$$A(z) = \sum_{n=0}^{\infty} \frac{z^{kn}}{[(\lambda+1)(\lambda+2)\dots(\lambda+n)]^{\kappa}}, \qquad k \geq 1,\ \lambda \neq -1, -2, \dots,$$

and of their derivatives.

In the main lemma 2 of that work the parameter s was indeterminate, enclosed within the limits from 1 to ν, and for any possible value of s one and the same prime number P_l was chosen. The assertion of this lemma is strengthened if the interval from 1 to ν is divided into several parts and, depending on the part in which s lies, one chooses one' s own prime number. The following is true.

Lemma 1. Let $\mu_1, \mu_2, \dots, \mu_r$ be natural numbers, $r \geq 1$, $0 = \mu_0 < \mu_1 < \dots < \mu_r = \nu$; let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ be real numbers, $0 \leq \varepsilon_i < 1/2$, $i = 1, 2, \dots, r$;

$$\varphi_\nu(z) = \sum_{n=0}^{\infty} a_{n,\nu} z^n, \qquad \nu = 0, 1, \dots, u; \tag{1}$$

$$f_\mu(z) = \sum_{n=}^{\infty} b_{n,\mu} z^n, \qquad \mu = 1, 2, \dots, v. \tag{2}$$

power series with coefficients from an algebraic field K, satisfying the conditions:

1. The series (1) are algebraically independent over the field of complex numbers.
2. For every natural t there exists an infinite set of systems of natural numbers $l_1, \dots, l_r$, $l_i \geq t$, $i = 1, 2, \dots, r$, such that for each value of i, $1 \leq i \leq r$:

a) the exact denominators of the numbers

$$b_{l_i,\mu_{i-1}+1},\ b_{l_i,\mu_{i-1}+2}, \dots,\ b_{l_i,\mu_i}$$

* For the definition of E-functions see ([1,4]).

contain the prime ideal $\mathfrak{p}_i$ from the field K in the powers respectively $k_{\mu_{i-1}+1}, k_{\mu_{i-1}+2}, \dots, k_{\mu_i}$, where $1 \leq k_{\mu_{i-1}+1} < k_{\mu_{i-1}+2} < \dots < k_{\mu_i}$;

b) the exact denominators of the numbers

$$a_{n,\nu}, \quad n = l_i - [\varepsilon_i l_i],\ l_i - [\varepsilon_i l_i] + 1, \dots, l_i + t, \quad \nu = 0, 1, \dots, u;$$

$$b_{n,\mu}, \quad n = l_i - [\varepsilon_i l_i],\ l_i - [\varepsilon_i l_i] + 1, \dots, l_i - 1, \quad \mu = 1, 2, \dots, v,$$

$$b_{l_i,1}, b_{l_i,2}, \dots, b_{l_i,\mu_i - 1}$$

may contain $\mathfrak{p}_i$ in powers less than $k_{\mu_{i-1}+1}$;

c) the exact denominators of the numbers

$$a_{n,\nu}, \quad n = [l_i/2],\ [l_i/2] + 1, \dots, l_i - [\varepsilon_i l_i] - 1, \quad \nu = 0, 1, \dots, u;$$

$$b_{n,\mu}, \quad n = [l_i/2],\ [l_i/2] + 1, \dots, l_i - [\varepsilon_i l_i] - 1, \quad \mu = 1, 2, \dots, v,$$

may contain $\mathfrak{p}_i$ in powers less than $\frac{1}{2} k_{\mu_{i-1}+1}$;

d) the exact denominators of the numbers

$$a_{n,\nu},\quad n=[l_i/2],\ [l_i/2]+1,\dots,l_i-[\varepsilon_i l_i]-1,\quad \nu=0,1,\dots,u;$$

$$b_{n,\mu},\quad n=[l_i/2],\ [l_i/2]+1,\dots,l_i-[\varepsilon_i l_i]-1,\quad \mu=1,2,\dots,v,$$

do not contain $\mathfrak{p}_i$.

Then the $u+v+1$ power series (1) and (2) are algebraically independent over the field of complex numbers.

Using the fundamental theorem of A. B. Shidlovskii ([1]), Lemma 1, and several auxiliary propositions on the distribution of prime numbers in various arithmetic progressions, one obtains a series of theorems formulated below on the transcendence and algebraic independence of values at algebraic points of certain hypergeometric E-functions that are solutions of linear differential equations of arbitrary orders with polynomial coefficients.

Let $\lambda,\lambda_1,\dots,\lambda_s$ be complex numbers; let $m_1,m_2,\dots,m_s$ be nonnegative integral rational numbers; $m=m_1+\dots+m_s\geq 1$. Denote $[\lambda,0]=1$, $[\lambda,n]=\lambda(\lambda+1)\cdots(\lambda+n-1)$, $n\geq 1$, and

$$A_{m,s}(z)=\sum_{n=0}^{\infty}\frac{1}{[\lambda_1+1,n]^{m_1}[\lambda_2+1,n]^{m_2}\cdots[\lambda_s+1,n]^{m_s}}\left(\frac{z}{m}\right)^{mn},\qquad (3)$$

$$\lambda_1,\dots,\lambda_s\neq 1,-2,\dots$$

Further, let $t_i=m_i$, $i=1,2,\dots,s-1$; let t_s be a natural number, $t_s\geq m_s$; $l=m_1+\dots+m_{s-1}+t_s$; $m_0=q_0=0$; $q_i=t_1+\dots+t_i$, $i=1,2,\dots,s$;

$$A_{m,s,\mu}(z)=1+\qquad (4)$$

$$+\sum_{n=1}^{\infty}\frac{(z/m)^{mn}}{[\lambda_1+1,n]^{m_1}\cdots[\lambda_{i-1}+1,n]^{m_{i-1}}[\lambda_i+1,n-1]^{m_i}\cdots[\lambda_s+1,n-1]^{m_s}(\lambda_i+n)^{\mu-q_{i-1}}}$$

$$\lambda_i\neq -1,-2,\dots;\quad \mu=q_{i-1}+1,q_{i-1}+2,\dots,q_i,\quad i=1,2,\dots,s.$$

In particular, for $s=2$ the functions (4) have the form

$$A_{m,2,\mu}(z)=1+\sum_{n=1}^{\infty}\frac{1}{[\lambda_1+1,n-1]^{m_1}[\lambda_2+1,n-1]^{m_2}(\lambda_1+n)^{\mu}}\left(\frac{z}{m}\right)^{mn},$$

$$\mu = 1, 2, \ldots, q_1,$$

$$A_{m,2,\mu}(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{[\lambda_1 + 1, n]^{m_1} [\lambda_2 + 1, n-1]^{m_2} (\lambda_2 + n)^{\mu - q_1}} \left(\frac{z}{m}\right)^{mn},$$

$$\mu = q_1 + 1, q_1 + 2, \ldots, l.$$

Theorem 1. Let λ_1, λ_2 be rational numbers, $\lambda_1, \lambda_2 \neq -1, -2, \ldots$, and, if one of them is an integer, let the other be different from half an odd number, and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,2,\mu}(\alpha)$, $\mu = 1, 2, \ldots, l$, are algebraically independent; 2) the m numbers $A_{m,2}(\alpha), A'_{m,3}(\alpha), \ldots, A^{(m-1)}_{m,2}(\alpha)$ are algebraically independent.

From Theorem 1, for $\lambda_1 = 0, \lambda_2 = \lambda, m_1 = m_2 = 1$, there follows the known result of C. L. Siegel ([4]) on the algebraic independence of the values of the functions $K_\lambda(z), K'_\lambda(z)$.

Theorem 1 also generalizes Theorem 1 of A. B. Shidlovskii from the paper ([5]) and refines the result of the latter. In Theorem 1 of the paper ([5]), which is obtained from Theorem 1 for $m_1 = m_2 = 1$ and $\lambda_1 = \mu, \lambda_2 = \lambda$, a restriction is imposed on the values of the parameters λ, μ, namely that the difference $\mu - \lambda$ be different from half an odd number. In our theorem this restriction is retained only in the case when one of the values λ or μ is an integer.

Theorem 1 contains Theorem 6 and, in part, Theorems 7, 8 of the paper ([3]).

Theorem 2. Let $\lambda_1, \lambda_2, \lambda_3$ be rational numbers, $\lambda_1, \lambda_2, \lambda_3 \neq -1, -2, \ldots$, let $\lambda_1 + \lambda_2$ be different from an integer, and let λ_3 and at least one of the numbers $2\lambda_1 - \lambda_2$ and $2\lambda_2 - \lambda_1$ be integers; moreover, if one of the numbers λ_1 and λ_2 is equal to $(2k_1 - 1)/2$, then the other is not equal to $k_2/4$, where k_1, k_2 are integers, and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,3,\mu}(\alpha)$, $\mu = 1, 2, \ldots, l$, are algebraically independent; 2) the m numbers $A_{m,3}(\alpha), A'_{m,3}(\alpha), \ldots, A^{(m-1)}_{m,3}(\alpha)$ are algebraically independent.

In the special case when $m_1 = m_2 = m_3 = 1$, $\lambda_1 = \lambda, \lambda_2 = \mu, \lambda_3 = 0$, the function $A_{3,3}(z)$ becomes the function $K_{\lambda,\mu}(z)$, considered by V. A. Oleinikov in the paper ([6]), where it is shown that for any rational values λ_1, λ_2 for which none of the numbers $|2\lambda_1 - \lambda_2|, |2\lambda_2 - \lambda_1|, |\lambda_2 - \lambda_1|$ is natural, the three numbers $A_{3,3}(\alpha), A'_{3,3}(\alpha), A''_{3,3}(\alpha)$ are algebraically independent for any algebraic value $\alpha \neq 0$.

Theorem 2 gives a certain refinement of this result. The numbers $A_{3,3}(\alpha), A'_{3,3}(\alpha), A''_{3,3}(\alpha)$ will also be algebraically independent when $2\lambda_1 - \lambda_2$ or $2\lambda_2 - \lambda_1$ is an integer, while $\lambda_1 + \lambda_2$ is different from an integer, but under the condition that if one of the numbers λ_1 or λ_2 is equal to $(2k_1 - 1)/2$, then the other is not equal to $k_2/4$, where k_1, k_2 are integers.

Theorem 3. Let $\lambda_i = b_i/a_i$, $\lambda_i \neq -1, -2, ...$, where a_i, b_i are integers, $a_i \geq 1$, $(a_i, b_i) = 1$, $i = 1, 2, \dots, s$, $s \geq 2$, $a_i a_j \neq 2$, $(a_i, a_j) = 1$, $1 \leq i < j \leq s$, and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,s,\mu}(\alpha)$, $\mu = 1, 2, \dots, l$, are algebraically independent; 2) the m numbers $A_{m,s}(\alpha), A'_{m,s}, \dots, A^{(m-1)}_{m,s}(\alpha)$ are algebraically independent.

Theorem 4. Let $\lambda_1, \dots, \lambda_s$ be rational numbers, $\lambda_1, \dots, \lambda_s \neq -1, -2, ...$, whose fractional parts satisfy the conditions

$$\{\lambda_1\} < \{\lambda_2\} < \dots < \{\lambda_s\}, \qquad 2\{\lambda_1\} > \{\lambda_{s-1}\},$$

and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,s,\mu}(\alpha)$, $\mu = 1, 2, \dots, l$, are algebraically independent; 2) the m numbers $A_{m,s}(\alpha), A'_{m,s}(\alpha), \dots, A^{(m-1)}_{m,s}(\alpha)$ are algebraically independent.

Put in expressions (3) and (4) $s = p + h$, where p, h are nonnegative integers.

Theorem 5. Let $\lambda_i = b_i/a_i$, $\lambda_i \neq -1, -2, ...$, where a_i, b_i are integers, $a_i \geq 1$, $(a_i, b_i) = 1$, $i = 1, 2, \dots, s$, $s \geq 2$; let A' and A'' be the least common multiples of the numbers, respectively, $a_1, a_2, \dots, a_s$ and $a_{p+1}, a_{p+2}, \dots, a_s$; let $(A', A'') = d$, $1 \leq d \leq 2$, let the fractional parts $\{\lambda_i\} > 1/2$ for $i \leq p$, $\{\lambda_i\} < 1/2$ for $i > p$, and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,s,\mu}(\alpha)$, $\mu = 1, 2, \dots, l$, are algebraically independent; 2) the m numbers $A_{m,s}(\alpha), A'_{m,s}(\alpha), \dots, A^{(m-1)}_{m,s}(\alpha)$ are algebraically independent.

In particular, for $h = 0$ or $p = 0$, Theorem 5 contains

Theorem 6. Let $\lambda_1, \dots, \lambda_s$ be rational numbers whose fractional parts, different from zero, are either greater than one half or less than one half, and let $\alpha \neq 0$ be any algebraic number.

Then: 1) the l numbers $A_{m,s,\mu}(\alpha)$, $\mu = 1, 2, \dots, l$, are algebraically independent; 2) the m numbers $A_{m,s}(\alpha), A'_{m,s}(\alpha), \dots, A^{(m-1)}_{m,s}(\alpha)$ are algebraically independent.

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Note: Figure translations are in progress. See original paper for figures.