EQUATIONS FOR FINITE-DIMENSIONAL PROBABILITY DISTRIBUTIONS OF A TURBULENCE FIELD

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Abstract

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EQUATIONS FOR FINITE-DIMENSIONAL PROBABILITY DISTRIBUTIONS OF A TURBULENCE FIELD

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We shall consider flows of an incompressible fluid with constant density $\rho =$ const. They are completely characterized by their solenoidal (i.e., divergence-free) velocity fields $\mathbf{u}(M)$, where $M = (\mathbf{x}, t)$ is a point of space-time; the pressure p can be expressed through the velocity field at the same instant of time by means of the formula

$$p(\mathbf{x}) = -\rho\Delta^{-1}(\mathbf{x}, \mathbf{x}')\,\frac{\partial^2 u_\alpha(\mathbf{x}')u_\beta(\mathbf{x}')}{\partial x'_\alpha \partial x'_\beta}, \tag{1}$$

where Δ^{-1} is the integral operator inverse to the Laplace operator (summation over repeated Greek indices is understood here and below). In the statistical description of turbulent flows, their velocity fields $\mathbf{u}(M)$ are regarded as random fields.

The analytically simplest (but least compact) statistical description of the field $\mathbf{u}(M)$ consists in specifying all moments $\langle u_{j_1}^{a_1}(M_1) \ldots u_{j_n}^{a_n}(M_n)\rangle$ of the random variables $u_{j_m} = u_j(M_m)$—the values of the velocity components on all possible finite sets of points $M_1, \ldots, M_n$, provided, of course, that all these moments exist (angle brackets here and below denote mathematical expectation). The moments satisfy dynamical (evolution) equations, the general method of deriving which from the Navier-Stokes equations was indicated by L. V. Keller and A. A. Friedman (^{2}). The equations for the moments are linear, but unclosed: in the equations for moments of order n, moments of order $n+1$ always occur.

The most compact (but analytically the most complicated) complete statistical description of the field $\mathbf{u}(M)$ consists in specifying its characteristic functional

$$\Phi[\vec{\theta}(M)] = \left\langle \exp\left\{ i \int \vec{\theta}(M)\mathbf{u}(M)\, dM \right\} \right\rangle. \tag{2}$$

It satisfies a dynamical equation, the derivation of which from the Navier–Stokes equations was indicated (in the case of a spatial functional) by E. Hopf (^{3}). Such an equation is linear and closed, but it contains variational derivatives, and no general methods for solving such equations are yet available.

A third method of complete statistical description of the field $\mathbf{u}(M)$, intermediate in compactness and analytical complexity, consists in specifying all finite-dimensional probability distributions for the values $\mathbf{u}_m = \mathbf{u}(M_m)$ of this field on all possible finite sets of points $M_1, \dots, M_n$. We shall characterize such distributions by the probability densities $P_{M_1,\dots,M_n}(\mathbf{u}_1, \dots, \mathbf{u}_n)$ or by their Fourier transforms with respect to $(\mathbf{u}_1, \dots, \mathbf{u}_n)$—the characteristic functions

$$\varphi_{M_1,\dots,M_n}(\vec{\theta}_1, \dots, \vec{\theta}_n) = \left\langle \exp\left\{ i \sum_{m=1}^{n} \vec{\theta}_m \mathbf{u}(M_m) \right\} \right\rangle . \tag{3}$$

Closed evolutionary equations for finite-dimensional probability distributions cannot be obtained, since the values $\mathbf{u}(\mathbf{x}_m, t)$ depend on the values of the initial velocity field $\mathbf{u}(\mathbf{x}, 0)$ not only at the points $\mathbf{x}_1, \dots, \mathbf{x}_n$, but also throughout the entire continuous space. But it is possible to obtain dynamical equations expressing, for example, the time derivatives of n-point characteristic functions in terms of the values both of these functions themselves and of certain $(n+1)$-point characteristic functions. The construction of such equations is the aim of the present note.

Let, for simplicity, all the points $\mathbf{x}_1, \dots, \mathbf{x}_n$ and all the time instants $t_1, \dots, t_n$ be distinct. Then, expressing in the formula

$$\frac{\partial}{\partial t_m} \varphi_{M_1,\dots,M_n}(\vec{\theta}_1, \dots, \vec{\theta}_n) = i\vec{\theta}_m \left\langle \frac{\partial \mathbf{u}(M_m)}{\partial t_m} \exp\left\{ i \sum_{m=1}^{n} \vec{\theta}_m \mathbf{u}(M_m) \right\} \right\rangle \tag{4}$$

the quantity $\partial \mathbf{u}(M_m)/\partial t_m$ in terms of the spatial derivatives of the field $\mathbf{u}(M_m)$ with the aid of the Navier–Stokes equations and formula (1), it is not difficult to arrive at the equations

$$\left(\frac{\partial}{\partial t_m} - i \frac{\partial^2}{\partial x_{m\alpha} \partial \theta_{m\alpha}} \right) \varphi_{M_1,\dots,M_n}(\vec{\theta}_1, \dots, \vec{\theta}_n) =$$

$$= -i\theta_{m\alpha} \Delta^{-1}(\mathbf{x}_m, \mathbf{x}) \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \left[\frac{\partial^2}{\partial \theta_\beta \partial \theta_\gamma} \varphi_{M_1,\dots,M_n M}(\vec{\theta}_1, \dots, \vec{\theta}_n, \vec{\theta}) \right]_{\vec{\theta}=0} +$$

$$+\nu \theta_{m\alpha} E(\mathbf{x}_m, \mathbf{x}) \frac{\partial^3}{\partial x_\beta \partial x_\beta} \left[\frac{\partial}{\partial \theta_\alpha} \varphi_{M_1,\dots,M_n M}(\vec{\theta}_1, \dots, \vec{\theta}_n, \vec{\theta}) \right]_{\vec{\theta}=0} , \tag{5}$$

where $M = (\mathbf{x}, t_m)$, and $E(\mathbf{x}_m, \mathbf{x})$ is the operator of replacing $\mathbf{x}$ by $\mathbf{x}_m$.

To equations (5) one may add the equations

$$\frac{\partial}{\partial x_{m\alpha}} \left[\frac{\partial}{\partial \theta_{m\alpha}} \varphi_{M_1,\dots,M_n}(\vec{\theta}_1, \dots, \vec{\theta}_n) \right]_{\vec{\theta}_m = 0} = 0, \tag{6}$$

which follow from the solenoidality of the velocity field. After the Fourier transformation with respect to $(\vec{\theta}_1, \dots, \vec{\theta}_n)$, from (5)–(6) one obtains the following equations for the probability densities of the finite-dimensional distributions:

$$\left(\frac{\partial}{\partial t_m} + u_{m\alpha} \frac{\partial}{\partial x_{m\alpha}} \right) p_{M_1,\dots,M_n}(\mathbf{u}_1, \dots, \mathbf{u}_n) =$$

$$= -\frac{\partial}{\partial u_{m\alpha}} \left[\Delta^{-1}(\mathbf{x}_m, \mathbf{x}) \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \int u_\beta u_\gamma p_{M_1,\dots,M_n M}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{u})\, d\mathbf{u} + \right.$$

$$\left. + \nu E(\mathbf{x}_m, \mathbf{x}) \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \int u_\alpha p_{M_1,\dots,M_n M}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{u})\, d\mathbf{u} \right]; \tag{7}$$

$$\frac{\partial}{\partial x_{m\alpha}} \int u_{m\alpha} p_{M_1,\dots,M_n}(\mathbf{u}_1, \dots, \mathbf{u}_n)\, d\mathbf{u}_m = 0. \tag{8}$$

For simultaneous characteristic functions or probability densities (when $t_1 = \dots = t_n = t$), the dynamical equations are obtained from (5) or (7) by summing over all m, taking into account the relation

$$\sum_m \frac{\partial}{\partial t_m} = \frac{\partial}{\partial t}.$$

Equations (5) and (7) are linear, but they are not closed. Apparently, they have not been published previously (only recently E. A. Novikov (1) obtained similar equations for the vorticity field). These equations are to some extent analogous to the well-known chain of equations for many-particle distribution functions in the kinetic theory of gases; it is possible that some of the approximate methods used to truncate that chain of equations may also prove useful in application to equations (5) or (7).

In the case of homogeneous turbulence one may put

$$\varphi_M(\vec{\theta}) = \varphi(\vec{\theta}, t); \qquad \varphi_{MM'}(\vec{\theta}, \vec{\theta}') = \varphi(\vec{\theta}, \vec{\theta}'; \mathbf{x}' - \mathbf{x}, t) \dots,$$

$$p_M(\mathbf{u}) = p(\mathbf{u}, t); \qquad p_{MM'}(\mathbf{u}, \mathbf{u}') = p(\mathbf{u}, \mathbf{u}'; \mathbf{x}' - \mathbf{x}, t) \ldots, \tag{9}$$

and equations (5) and (7) for $n = 1$ take the form

$$\partial\varphi(\vec{\theta}, t)/\partial t = -i\theta_\alpha \psi_\alpha(\vec{\theta}, t);$$

$$\psi_\alpha(\vec{\theta}, t) = \Delta^{-1}(0, \mathbf{r}) \frac{\partial^3}{\partial r_\alpha \partial r_\beta \partial r_\gamma} \left[\frac{\partial^2}{\partial\theta'_\beta \partial\theta'_\gamma} \varphi(\vec{\theta}, \vec{\theta}'; \mathbf{r}, t) \right]_{\vec{\theta}'=0} +$$

$$+ i\nu E(0, \mathbf{r}) \frac{\partial^2}{\partial r_\beta \partial r_\beta} \left[\frac{\partial}{\partial\theta'_\alpha} \varphi(\vec{\theta}, \vec{\theta}'; \mathbf{r}, t) \right]_{\vec{\theta}'=0}; \tag{10}$$

$$\partial p(\mathbf{u}, t)/\partial t = -\partial q_\alpha(\mathbf{u}, t)/\partial u_\alpha;$$

$$q_\alpha(\mathbf{u}, t) = \Delta^{-1}(0, \mathbf{r}) \frac{\partial^3}{\partial r_\alpha \partial r_\beta \partial r_\gamma} \int u'_\beta u'_\gamma p(\mathbf{u}, \mathbf{u}'; \mathbf{r}, t)\, d\mathbf{u}' +$$

$$+ \nu E(0, \mathbf{r}) \frac{\partial^2}{\partial r_\beta \partial r_\beta} \int u'_\alpha p(\mathbf{u}, \mathbf{u}'; \mathbf{r}, t)\, d\mathbf{u}', \tag{11}$$

where $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. In the case of isotropic turbulence $\varphi_M(\vec{\theta})$ will depend only on two arguments $\theta = |\vec{\theta}|$ and t; $p_M(\vec{\theta})$—on $u = |\mathbf{u}|$ and t; $\varphi_{MM'}(\vec{\theta}, \vec{\theta}')$—not on 13 arguments $(\mathbf{x}, \mathbf{x}', t, \vec{\theta}, \vec{\theta}')$, but only on 7 arguments $r = |\mathbf{r}|$, $\theta = |\vec{\theta}|$, $\theta' = |\vec{\theta}'|$, $\mathbf{r} \cdot \vec{\theta}$, $\mathbf{r} \cdot \vec{\theta}'$, $\vec{\theta} \cdot \vec{\theta}'$, and t; and analogously $p_{MM'}(\mathbf{u}, \mathbf{u}')$—only on 7 arguments $r = |\mathbf{r}|$, $u = |\mathbf{u}|$, $u' = |\mathbf{u}'|$, $\mathbf{r} \cdot \mathbf{u}$, $\mathbf{r} \cdot \mathbf{u}'$, $\mathbf{u} \cdot \mathbf{u}'$, and t. Equations (10) and (11) then take the form

$$\partial\varphi(\theta, t)/\partial t = -i\psi(\theta, t); \qquad \psi(\theta, t) = \theta_\alpha \psi_\alpha(\vec{\theta}, t); \tag{12}$$

$$\partial p(u, t)/\partial t = -\frac{1}{u^2} \frac{\partial}{\partial u} [u q(u, t)]; \qquad q(u, t) = u_\alpha q_\alpha(\mathbf{u}, t). \tag{13}$$

All the equations written here can be generalized without particular difficulty to problems on the turbulent motion of a fluid in a field of random external forces.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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